A Note on E. Rogge's and P. Schoenbucher's Article "Modelling Dynamic Portfolio Credit Risk"

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Introduction. Ebbe Rogge and Philipp J. Schoenbucher recently presented a new class of credit basket models based upon a generalised class of Archimedean copula functions (**[RS]**). Their method applies techniques developed in an earlier contribution of Philipp J. Schoenbucher and Dirk Schubert (**[SS]**). The approach is of great theoretical and practical importance, since the *hidden* credit spread or default intensity dynamics of copula-driven credit basket models are disclosed and analysed. Moreover, from a practitioner's point of view their approach can be *inverted* and a copula can be constructed or calibrated – within a certain family – via the implied credit spread widening of surviving assets upon default of an arbitrary asset in the basket. It should be highlighted that their model shows a very desirable *updating behaviour*, which is not shared by standard models such as the Gaussian copula or the Student-*t*-copula.

The purpose of our note is to show a correspondence between the generalised Archimedean copulas introduced by Rogge and Schoenbucher and a family of credit basket models based upon conditional independence and a frailty approach (for a general reference of conditional independence models we refer the reader to Bielecki's and Rutkowski's text book **[BR]**).

Notation. Since we want to keep this note short and self-contained, the notation is kept as lean as possible. Vectors of (random) variables are denoted by $\overline{\ldots}$. A basket of n different, not yet defaulted assets is considered, where the default time of the i-th asset is denoted by the standard symbol τ_i . The credit default models are set in a filtered probability space, for which the usual conditions are assumed and the filtration will incorporate the observed market and credit default behaviour. The joint survival distribution of the asset basket is defined as (with regard to the initial market and credit information):

$$S(t_1,\ldots,t_n) = prob[\tau_1 > t_1,\ldots,\tau_n > t_n] \qquad \text{with } t_i \geq 0 \text{ for } i = 1,2,\ldots,n$$

Construction of credit basket models. The Conditional Independence Frailty Model (CIF) and the Generalised Archimedean Copula (GAC) are briefly sketched in terms of their defining model parameters and the description is concluded with a statement of the corresponding joint survival distribution and an interpretation, which is exhaustive for the model description.

<u>A. Conditional Independence Frailty Model</u> $CIF(\overline{\mu},\overline{X})$

1. Model parameters

- $\mu_i : [0,\infty[\to [0,\infty[$ denote strictly monotone increasing and unbounded functions for $i = 1,2,\ldots,n$ with $\mu_i(0) = 0$
- X_i denote positive, generally dependent random variables for $i = 1,2,\ldots,n$ with associated Laplace transforms $\varphi_{X_i}(z) = E_{X_i}\left[e^{-z \cdot X_i}\right]$

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2. Definition of joint survival distribution

$$S^{CIF(\overline{\mu},\overline{X})}(t_1,\ldots,t_n) = E_{\overline{X}}\left[e^{-\sum_{i=1}^{n}\mu_i(t_i)\cdot X_i} \right]$$

3. Interpretation & comments

The wording conditional independence is due to the assumed model property that upon a realisation of the random vector $\overline{X}$ independence of the default times follows. The marginal distributions are determined by:

$$prob[\tau_i > t_i] = E_{X_i}\left[e^{-\mu_i(t_i)\cdot X_i}\right] = \varphi_{X_i}\left[\mu_i(t_i)\right]$$

It probably might be helpful if the reader interprets the product $\mu_i(t_i)\cdot X_i$ as an integral over (pseudo-) intensities, e.g., in the form of $\mu_i(t_i)\cdot X_i = \int_0^{t_i}\lambda_i(s,X_i)\,ds$, which allows the description of the marginal distribution as

$$prob[\tau_i > t_i] = E_{X_i}\left[e^{-\int_0^{t_i}\lambda_i(s,X_i)\,ds} \right]$$

The random variables X_i play the role of (latent) risk factors, which affect the default intensities of the assets in the basket. In practice, the random variables are typically decomposed in the form $X_i = \sum_{j=1}^{m} w_{ij}\cdot Z_j$ with positive weights w_{ij} and independent, positive random variables Z_j, which can describe macroeconomic, sector-related or residual, asset-specific risk drivers. Therefore, the dependence behaviour is not so much credit event-driven, but more intuitively by the risk of a simultaneous credit spread widening or tightening due to common risk drivers or *shared frailties*.

B. Generalised Archimedean Copula $GAC(\overline{S},\overline{Y})$

1. Model parameters

- ▶ $S_i : [0,\infty[\to [0,1]$ denote survival functions with $S_i(0)=1$ for $i=1,2,\ldots,n$, i.e. $1-S_i : [0,\infty[\to [0,1]$ define random distributions
- ▶ Y_i denote positive, generally dependent random variables for $i=1,2,\ldots,n$ with associated Laplace transform $\varphi_{Y_i}(z) = E_{Y_i}\left[e^{-z\cdot Y_i}\right]$; the inverse $\varphi_{Y_i}^{-1}$ of the Laplace transforms exists because of $\frac{\partial\varphi_{Y_i}}{\partial z} < 0$ (note, that here the positive-ness of the Y_i is exploited)

2. Definition of joint survival distribution

$$S^{GAC(\overline{S},\overline{Y})}(t_1,\ldots,t_n)=E_{\overline{Y}}\left[e^{-\sum_{i=1}^{n}\varphi_{Y_i}^{-1}[S_i(t_i)]\cdot Y_i}\right]$$

Further details and the proof of the existence of such a distribution can be found in Rogge and Schoenbucher (**[RS]**, Proposition 9), although formulated in the framework of survival copulas and based on a slightly different notation. This set-up permits a richer definition of default dependencies, e.g., asymmetric dependencies can be introduced, and generalises the class of Archimedean copulas which are described by

$$S^{Archimedean}(u_1,\ldots,u_n)=E_W\left[e^{-\sum_{i=1}^{n}\varphi_W^{-1}[u_i]\cdot W}\right]=\varphi_W\left(\sum_{i=1}^{n}\varphi_W^{-1}[u_i]\right)$$ for a positive random variable W

Actually, the authors have started with a decomposition $Y_i=\sum_{j=1}^{m}w_{ij}\cdot Z_j$ using positive weights w_{ij} and independent, positive random variables Z_j (compare with our comment on the Conditional Independence Frailty model for a similar decomposition). This yields a nicer description of the joint survival distribution due to the following identity:

$$S^{GAC(\overline{S},\overline{Y})}(t_1,\ldots,t_n)=E_{\overline{Y}}\left[e^{-\sum_{i=1}^{n}\varphi_{Y_i}^{-1}[S_i(t_i)]\cdot Y_i}\right]=E_{\overline{Z}}\left[e^{-\sum_{i=1}^{n}\varphi_{Y_i}^{-1}[S_i(t_i)]\cdot\left[\sum_{j=1}^{m}w_{ij}\cdot Z_j\right]}\right]=$$

$$=E_{\overline{Z}}\left[e^{-\sum_{j=1}^{m}Z_j\cdot\sum_{i=1}^{n}\varphi_{Y_i}^{-1}[S_i(t_i)]\cdot w_{ij}}\right]=\prod_{j=1}^{m}E_{Z_j}\left[e^{-Z_j\cdot\sum_{i=1}^{n}\varphi_{Y_i}^{-1}[S_i(t_i)]\cdot w_{ij}}\right]=\prod_{j=1}^{m}\varphi_{Z_j}\left[\sum_{i=1}^{n}\varphi_{Y_i}^{-1}[S_i(t_i)]\cdot w_{ij}\right]$$

Statement of the main result.

The Conditional Independence Frailty model and the Generalised Archimedean Copula model both support the analytical tractability of credit default dynamics. In particular they allow a nice characterization of credit spread widening effects of surviving assets upon default of another asset in the basket, which can be used for calibration purposes and analysis of hedging strategies.

If the random variables X_i respectively Y_i have simple Laplace transforms and belong to a family of summation-stable distributions, e.g. Gamma distributions, basket loss distributions can be efficiently (<< 1 sec) evaluated via Fast Fourier Transform or recursive schemes a la Panjer (see for instance the text book of Rolski et. al **[RSST]**, chapter 4).

The two model classes have more properties in common, which are extensively discussed for the Generalised Archimedean Copula in the paper **[RS]**. This is not a surprise, because they are actually equivalent model descriptions.

Correspondence of default basket models: *There exists a one-to-one correspondence between the CIF and GAC model universe, to be more precise the (bijective) mappings can be defined in terms of the model input parameters as follows:*

$$\begin{array}{ccc} CIF & \to & GAC \\ \left(\overline{\mu},\overline{X}\right) & \mapsto & \left(\overline{\varphi_X \circ \mu},\overline{X}\right) \end{array} \qquad \begin{array}{ccc} GAC & \to & CIF \\ \left(\overline{S},\overline{Y}\right) & \mapsto & \left(\overline{\varphi_Y^{-1} \circ S},\overline{Y}\right) \end{array}$$

where the vector function $\overline{\varphi_X \circ \mu}$ *(respectively* $\overline{\varphi_Y^{-1} \circ S}$ *) is defined (coordinate-wise) by* $\left(\overline{\varphi_X \circ \mu}\right)_i(t) = \varphi_{X_i}\left[\mu_i(t)\right]$ *(respectively by* $\left(\overline{\varphi_Y^{-1} \circ S}\right)_i(t) = \varphi_{Y_i}^{-1}\left[S_i(t)\right]$*).*

Proof of the correspondence:

As for the mapping $CIF \to GAC$: The mapping is well-defined since by definition $S_i = \varphi_{X_i} \circ \mu_i$ defines a survival function for $i = 1,2,...,n$. Note, that the inverse function of φ_{X_i} exists because of $\frac{\partial \varphi_{X_i}}{\partial z} < 0$. If we take an arbitrary test vector of dates for the evaluation of the joint survival distribution, direct calculation yields:

$$S^{CIF\left(\overline{\mu},\overline{X}\right)}\left(t_1,...,t_n\right) = E_{\overline{X}}\left[e^{-\sum_{i=1}^{n}\mu_i(t_i)\cdot X_i}\right] = E_{\overline{X}}\left[e^{-\sum_{i=1}^{n}\varphi_{X_i}^{-1}\left[S_i(t_i)\right]\cdot X_i}\right] = S^{GAC\left(\overline{S},\overline{X}\right)}\left(t_1,...,t_n\right) =$$

$$= S^{GAC\left(\overline{\varphi_X \circ \mu},\overline{X}\right)}\left(t_1,...,t_n\right)$$

Since the date vector $\left(t_1,...,t_n\right)$ was arbitrarily chosen, we have proven $S^{CIF\left(\overline{\mu},\overline{X}\right)} = S^{GAC\left(\overline{\varphi_X \circ \mu},\overline{X}\right)}$.

Similarly, applying the inequality $\frac{\partial\left(\varphi_{Y_i}^{-1} \circ S_i\right)}{\partial t} > 0$ the inverse mapping $GAC \to CIF$ is validated. This completes the proof. ■

References

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