The impact of Stock returns volatility on credit default swap rates: A copula study

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Abstract

The aim of this paper is to study the impact of Stock returns volatility of reference entities on credit default swap rates using a new dataset from the Japanese market. The majority of empirical research suggests the inadequacy of multinormal distribution and then the failure of methods based on correlation for measuring the structure of dependency. Using a copula approach, we can model the different relationships that can exist in different ranges of behavior. We study the bivariate distributions of credit default swap rates and the measure of stock return volatility estimated with GARCH (1,1) and focus on one parameter Archimedean copula. Starting from the empirical rank correlation statistics (Kendall's tau and Spearman's rho), we estimate the parameter values of each copula function presented in our study. Then, we choose the appropriate Archimedean copula that better fit to our data. We emphasize the finding that pairs with higher rating present a weaker dependence coefficient and then, the impact of stock return volatility on credit default swap rates is higher for the lowest rating class.

Key words: Copulas functions; credit default swap; volatility; bivariate distribution; Non-parametric estimation, Semi-parametric estimation.

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1. Introduction

One of the main issues in finance and insurance is the aggregation of individual risks. The problem can be solved by assuming that the individual risks represented by random variables are independent or are only dependent by a common factor. The problem is hardly posed when we want to model the joint distribution of dependent individual risks. Then, the knowledge of dependence structure between financial assets or claims is a crucial step to achieve adequate risk management in finance and insurance. Again, the problem can be solved by assuming a Gaussian behaviour of the vector of risks with some given covariance matrix. However, the dimension of risk captured by the correlation matrices is only satisfying for elliptic distributions. Embrechts et al (1999) show the necessity to leave the Gaussian world[1] since the normal joint distribution cannot catch some key futures of the dependence like the tail dependence and the classic correlation coefficient is only adapted for assessing linear dependence and can lead to a very strong underestimation of the real incurred risk. Then, the multivariate normal distribution is not a good model for the joint distribution of many economic variables. Many dependency measures have been proposed according to concepts such as concordance, quadrant dependency, etc. In the case of two random variables, structure of dependency can have a long variety of forms according to some specifications. Most popular examples are based on the concept of concordance and discordance which are scale invariant measures such as Kendall's tau and the Spearman's rho. More recently, the introduction of the theory of copulas in finance by Embrechts et al (1999) has had a great impact in the study of dependence of random variables. In this paper, we use Archimedean copulas to model dependency between credit default swap prices and stock return volatility in view of the construction of bivariate distributions based on that dependency structure. Volatility is a statistical measure of the tendency of a market or security to rise or fall sharply within a short period of time. Volatile markets are characterized by wide price fluctuations and heavy trading. They are caused by things like company news, a recommendation from a well known analyst or unexpected earnings results.

Recent empirical work has been done on credit derivative markets. Hull, Predescu & White (2004) analyze the impact of credit rating announcements in the pricing of credit default swap. Norden & Weber (2004) analyse the empirical relationship between credit default swap, bond and stock markets. They examine weekly and daily stock lead-lag relationship in a vector autoregressive model and the adjustment between markets caused by cointegration. They find that weekly and daily stock returns are negatively associated with credit default swap and bond spread changes. Also, the sensitivity of the credit default swap market to prior stock market movements is significantly related to the firm's average credit worthiness. Ericsson, Jacobs & Oviedo (2004) investigate the relationship between theoretical determinants of default risk (firm leverage, volatility and the riskless interest rate) and

[1] The first paper that report the non-normality of the distribution of many interesting economic variables was: Mandelbrot, B. (1963), "The variation of certain speculative prices", Journal of Business, 26, p.394-419.

actual market premia of credit default swap using linear regression. They find that a 1% increase in annualzed equity volatility raises the credit default swap premia by 1 to 2 basis points.

In our paper, we analyse the impact of stock return volatility estimated with GARCH (1,1) on the rates of credit default swap with copula study. We adopt nonparametric approach to estimate copulas parameters and then we expose a manner to choose the appropriate copula that better fit to data and model the joint distribution of credit default swap rates and stock return volatility. The theory of copula dates back to Sklar (1959). The copula function links the univariate margins with their full multivariate distribution. It presents a useful tool when modelling non Gaussian data since the Pearson's correlation coefficient is adapted for linear dependence and normal distribution. One appealing feature of a copula function is that the margins do not depend on the choice of the dependency structure and then, we can model and estimate the structure of dependency and the margins separately. Copula function describes the whole multivariate distribution and present the property that it is invariant under strictly increasing transformations of the margins. Also, they present a basis for flexible techniques for simulating dependent random vectors.

Despite their long history in statistics, copulas functions have been applied in many areas in finance and insurance literature only very recently, but research in this area grows rapidly. Hull & white (1998) and Embrechts et al. (2003) study the portfolio Value-at-Risk (VAR) with copula. Micocci & Masal (2004) uses copula functions to model dependence structure among stock returns with the aim of calculating the Value-at-Risk of a stock portfolio. Costinot et al (2000) study the extreme dependence between international equity markets and analyse the East Asian crisis with copulas functions. Longin & Solnik (2001) use a Gumbel copula to estimate the extreme correlations across international equity markets. Mashal & Zeevi (2002) investigate the extreme co-movements between financial assets (equities, currencies and commodities) by testing the underlying dependence, a t-dependence structure, via copula. Hu (2003) estimate the dependence structure in several major stock markets using a mixed copula approach. Bouyé & Salmon (2002) provide an application of copula in capturing dependence between financial assets that follow non-Gaussian distributions and then for modelling credit risk, pricing option and portfolio design. Coutan et al (2001) define multivariate risk-neutral distributions with copulas and derive pricing formulas for some multi-asset options. One of the most popular application of copulas is credit risk modelling. Li (2000) and Frey & McNeil (2002) analyse default correlation and the pricing and hedging of credit sensitive instruments with copulas functions. Jouanin et al (2001) address the problem of incorporating default dependency in intensity-based credit risk models by using copulas functions to model the joint distribution of the default times. They use two different approaches: the first is the survival approach following the work of Schönbucher & Schubert (2001) and the second is the threshold approach which they model dependence between the triggers of the firm. Mashal & Naldi (2001) present a methodology for estimation, simulation and pricing of multiname credit derivatives. The dependence structure is modelled by a t-copula. Accioly & Chiyoshi (2004) use copula functions in oil fields development plans and petroleum engineering. They analyse the relationship between drilling durations and measured depth. In their study, the Clayton copula was confirmed by non parametric and semi-parametric procedures. The main problem of copula approach is the existence of an infinite number of possible copulas and actually, no general empirical research has determined exactly.

The remainder of this paper is organised as follows: section two describes some mathematical background about the concept of copula and its properties. In section three, we illustrate different methods of parameters estimation of copula. Section four describes our data, methodology and results to detect the possible relationship between credit default swap rates and stock return volatility. Section five concludes the paper.

2. Copulas functions

Copula was first used in survival analysis and actuarial sciences. Fisher (1997) shows the importance of copulas to statisticians as a way of studying scale-free measures of dependence and as a tool to construct families of bivariate distributions with given margins.

For n uniform random variables $u_1, u_2, ..., u_n$, the joint distribution function C is defined as:

$C(u_1, u_2, ..., u_n, \theta) = \Pr[U_1 \le u_1, U_2 \le u_2, ..., U_n \le u_n]$ where θ the dependence parameter.

As we only need the concept of copulas for two dimensions, we present the following definition:

A copula function is the restriction to $[0,1]^2$ of a continuous bivariate distribution function whose margins are uniform on $[0,1]$. A (bivariate) copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies the boundary conditions $C(t,0) = C(0,t) = 0$ **and** $C(t,1) = C(1,t) = t$ **for** $t \in [0,1]$.

Similarly, copula satisfies the 2-increasing property: $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0$ for all u_1, u_2, v_1, v_2 **in** $[0,1]$ **and** $u_1 \le u_2$ **and** $v_1 \le v_2$.

A copula is symmetric if $C(u,v) = C(v,u)$ **for all** (u,v) **in** $[0,1]^2$ and is asymmetric otherwise.

Sklar (1959) shows the importance of copulas as a universal tool for studying multivariate distributions[2].

Theorem 1: Let F be a multivariate n-dimensional distribution function with marginals $F_1,\ldots,F_n$. then it exists a copula such that $F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n))$; $(x_1, ..., x_n \in \Re)$.

If the marginal distributions $F_1,\ldots,F_n$ are continuous, then C is unique.

By definition, applying the cumulative distribution function (CDF) to a random variable (r.v.) results in a r.v. that is uniform on the interval [0, 1]. Let X a random variable with continuous distribution function F_X, $F_X(X)$ is uniformly distributed on the interval [0,1]. This result is known as the probability integral transformation theorem and present many statistical procedures. With this result in hand, we

[2] The original definition of copula is given by Sklar (1959) and the Sklar's theorem is considered as the most important theorem about copula functions. The problem of obtaining a joint distribution is reduced to selecting the appropriate copula.

may introduce the copula using basic statistical theory. In particular, the copula C for (X,Y) is just the joint distribution function for the random couple $F_X(X)$, $F_Y(Y)$ provided F_X and F_Y are continuous.
The previous representation is called canonical representation of the distribution. Thus, copulas link joint distribution functions to their margins. Then, in continuous distribution, the problem of obtaining the joint distribution has reduced to selecting the appropriate copula. We can build multidimensional distributions with different marginals.
Numerous copulas can be found in the literature (see Nelson (1999) and Joe (1997)). The most commonly applied copula function (especially in finance modelling) is the normal copula[3]. This could be justified by the fact that the multivariate normal distribution has two appealing characteristics: first, their marginals distributions are normal and second, it can be fully described by their marginal distribution and a variance-covariance matrix. For univariate margins $F_1,\ldots,F_n$ which are Gaussians, the dependence structure among the margings is described by a unique normal copula function defined by: $C(u_1,\ldots,u_n) = N_n\left[N^{-1}(u_1),\ldots,N^{-1}(u_n);\Gamma\right]$

Where $N_n(.;\Gamma)$ denotes the cumulative standard normal distribution function with linear correlation matrix R and N^{-1} is the inverse of the standard univariate Gaussian distribution function.

Archimedean copula class

Let Φ denote a function $\Phi:[0,1]\rightarrow[0,\infty]$ which is continuous and satisfies:

- $\Phi(1)=0$
- $\Phi(0)=\infty$
- For all $t\in(0,1)$, $\Phi'(t)<0$, then Φ is decreasing.
- For all $t\in(0,1)$, $\Phi''(t)\geq 0$, then Φ is convex.

The function Φ has an inverse $\Phi^{-1}:[0,\infty]\rightarrow[0,1]$ which has the same properties except that $\Phi^{-1}(0)=1$ and $\Phi^{-1}(\infty)=0$. A copula is said to be an Archimedean copula if its distribution function can be written as follows: $C_{arch}(u,v)=\Phi^{-1}[\Phi(u)+\Phi(v)]$, for all $0\leq u,v\leq 1$ and Φ is called a generator function of copula that satisfies the following properties:
- C is symmetric; i.e., $C(u,v))=C(v,u)$ for all $u,v\in[0,1]$.
- C is associative; i.e., $C(C(u,v),w)=C(u,C(v,w))$ for all $u,v,w\in[0,1]$.
- if $k>0$ is any constant, then $k\Phi$ is also generator of Φ .
The Archimedean copula has simplified the construction of bivariate distributions and it has many families that are capable to present different structure of dependency and different methods are

[3] Credit Metrics ™ and KMV model implicitly incorporate copula functions based on the multivariate Gaussian distribution of asset value process.

developed to estimates its parameters. We only need to find functions which will serve as generators and define the corresponding copula. The important result about Archimedean copula have been presented, among others, by Genest & Rivest (1993) and Genest et al (1995) who studied statistical inference of copulas parameters. Frees & Valdez (1998) showed their usefulness in actuarial sciences field. Müller & Scarsini (2003) considered several notions of positive dependence in the case of Archimedean copula. The most common Archimedean copula has closed form expression. We present examples of bivariate Archimedean copulas[4].

The independent copula

Let X and Y two continuous random variables with marginal distributions F_1 and F_2 and joint distribution F .assume that X and Y are independent, then: $F(x,y) = F_1(x).F_2(y)$, $\forall x, y \in [-\infty, +\infty]$. Suppose that $\Phi(t) = -\ln(t)$ for $t \in [0,1]$. This is a decraesing convexe function, satisfying $\Phi(0) = \infty$. Generating C from the expression of the Archimedean copula, we obtain

$$C(u,v) = \exp(-[(-\ln u) + (-\ln v)]) = u.v = \prod(u,v), \text{ and } F(x,y) = \prod(F_1(x), F_2(y)).$$

Then, the product or the independent copula is an Archimedean copula.

$$C_{ind}(u_1, u_2) = u_1 \times u_2 \text{ for all } (u_1, u_2) \textbf{ in } [0,1]^2.$$

Clayton copula

This family proposed by Clayton (1978) is the following:

Let $\Phi(t) = \frac{(t^{-\theta} - 1)}{\theta}$ with $\theta \in [-1, \infty)/\{0\}$, then $C_\theta^{clayton}(u,v) = \max\left[\left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}, 0\right]$

If $\theta > 0$, then $\phi(0) = \infty$ and we can simplify the above expression: $C_\theta^{clayton}(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}$, θ expresses the degree of dependence among the marginal components.

Gumbel Copula

This family proposed by Gumbel (1960) is the following:

Let $\Phi(t) = (-\ln t)^\theta$, with $\theta \geq 1$.

$$C_\theta^{Gumbel}(u,v) = \exp\left(-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{\frac{1}{\theta}}\right); \quad 0 \leq u, v \leq 1 \quad .$$

Where $\theta \in [1, \infty)$ controls the degree of dependence between u and v . $\theta = 1$ would mean independent variables and $\theta \to \infty$ means perfect dependence.

[4] See Appendix 1.

Frank copula[5]

This family is proposed by Frank (1979) as follows:

Let $\Phi(t) = -\ln \frac{-e^{-\theta t}-1}{e^{-\theta}-1}$ with $\theta \in \Re / \{0\}$

$$C_{\theta}^{Frank}(u,v) = -\frac{1}{\theta}\ln\left(1+\frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{\left(e^{-\theta}-1\right)}\right).$$

3. Parameters estimation

A basic assumption in our paper is that the data are suitably modelled by an Archimedean copula. Then, a general estimation procedure for Archimedean copula should be of interest.
In this section, we will discuss nonparametric (Genest & Rivest (1993)), parametric (Likelihood) and semi-parametric (Genest et al (1995)) methods of estimating Archimedean copula parameters.

3.1. Nonparametric estimation

Much effort has gone into developing procedures that can be used in the absence of strong a priori restrictions. This effort examines nonparametric methods which do not impose parametric restrictions on functional form. Genest & Rivest (1993) suggested a nonparametric method for estimating the dependence function of a pair of random variables under the assumption that their uniform representation is Archimedean. Their method relies on the estimation of the univariate distribution function associated with the probability integral transformation and requires complete data. The best fitting Archimedean model is the one whose probability integral transformation distribution is the closest to its empirical estimate. The bivariate probability integral transformation of (X,Y) with joint distribution function H is defined as $V = H(X,Y)$. It is not generally true that the distribution function K of V is uniform on [0,1] even if H is continuous. Similarly, K does not characterize H since K does not contain any information about the marginals F_X and F_Y.
The problem of specifying a probability model for independent observations $(x_1,y_1),\ldots,(x_n,y_n)$ from a bivariate non normal distribution function H(X,Y) can be simplified by expressing H in terms of its marginals F_X and F_Y and its associated dependence function C.
Then, Archimedean copulas are characterized by the stochastic behaviour of the random variate $V = H(X,Y)$. The univariate distribution function is defined as:
$K(v) = \Pr[H(X,Y) \le v] = \Pr[C\{F_X(X), F_Y(Y)\} \le v]$ on the interval (0,1). The estimation of K can be accomplished in two steps: they construct the empirical bivariate distribution $H_n(X,Y)$ and they compute $H_n(x_i, y_i)$ for $i = 1,\ldots,n.$ and use those pseudo observations to construct one-dimensional empirical distribution function for K.

[5] The properties of Frank's family of bivariate distributions are studied in the article of Christian Genest (1987) "Frank's family of bivariate distributions", Biometrika, 74, 3, p. 549-555.

The Archimedean copula presents an appealing property: each copula has an analytical expression that links its parameters to its related Kendall tau. Here, we present the important theorem in the theory of Archimedean copula (Genest & MacKay, 1986):

Theorem: Let (X,Y) be a pair of random variables whose distribution H is of the form $\left[C_\Phi(x,y)=\Phi^{-1}\{\Phi(x)+\Phi(y)\}\right]$ for some Φ, then: $\tau = 4\int_0^1 \frac{\Phi(t)}{\Phi'(t)}dt+1$.

Then, we can estimate the parameter from the parametric copula using a relationship between the Kendall's τ and the Archimedean copula.

3.2. Semi-parametric estimation

Genest et al (1995) proposed a semi-parametric procedure for estimating the dependence parameters in a family of multivariate distributions when one does not want to specify any parametric model to describe the marginal distribution. This procedure consists of transforming the marginal observations into uniformly distributed vectors using the empirical distribution function. Then, the copula parameters are estimated by maximisation of a Pseudo log-likelihood function.

When nonparametric estimates are contemplated for the marginals, inference about the dependence parameter must be margin free. Given a random sample $\{(X_{1k},...,X_{pk}):k=1,...,n\}$, from distribution $F_\alpha(x_1,...,x_p)=C_\varepsilon\{F_1(x_1),...,F_p(x_p)\}$. In the construction of the likelihood function, we will be interested to the parametric representation of the copula, specifically, the copula density. The procedure consists of selecting the parameter value $\hat{\alpha}_n$ that maximises the pseudo log-likelihood: $L(\alpha)=\sum_{k=1}^{n}\log\left[c_\alpha\{F_{1n}(X_{1k}),...,F_{pn}(X_{pk})\}\right]$ in which c_α is the copula density and F_{in} is the rescaled empirical distribution function given by: $F_{in}(x)=\frac{1}{n+1}\sum_{j=1}^{n}1(X_{ij}\le x)$ for any $1\le i\le p$. Genest et al (1995) examined the statistical properties of the proposed estimator. It is shown that it is consistent, asymptotically normal and fully efficient at independence.

4. Empirical study

Starting from the assumption that the Archimedean dependence structure is appropriate (an assumption that we will retain throughout this work), our empirical study consists to detecting the possible relationship of credit default swap rates and stock return volatility of the corresponding reference entity through Archimedean copula. We restrict our attention to one parameter copula and then, we can distinguish between modelling the marginal distribution of the data and estimating the parameter θ of the copula family. We adopt nonparametric approach of Genest & Rivest (1993) that not consider the marginal distribution function of each data set (X and Y). The parameter θ of the copula is estimated using Kendall’s rank correlation. Once we have estimated θ , we choose the appropriate copula that fits best our bivariate data.

4.1. Data description and methodology

Our daily data are issued from Tokyo International Financial futures Exchange Inc. (TIFFE) and cover the period going from 26 March 2004 to 22 December 2004 (we have 183 observations per series). It consists of Japan credit default swap rates, expressed on basis point and sorted by rating and industry name. The ratings are established by JCR (Japan Credit Rating). A credit default swap rate is the singular rate calculated from the best indicative rates in the over the counter market provided by financial institutions. The indicative rates provided by the designed financial institutions are the rates (premium) of the following standardized credit default swap contracts:

- 5 year maturity.
- Amount of 500 million yen for notional principle.
- Three credit event such as “Bankruptcy”, “Failure to pay” and “Restructuring”.
- A physical settlement is required in case that any credit event occurs.

The credits that underlying default swaps are composed with 43 contracts and are listed by rating[6] and economic sector:

Rating	Number of credit
AAA	7
AA^+	3
AA	3
AA^-	7
A^+	8
A	6
A^-	3
BBB^+	5
BBB^-	1

Economic sector		Number of credit
Electric Products	(EP)	7
Foods	(F)	2
Land Transportation	(LT)	3
Airlines	(A)	2
Real Estate	(RE)	2
Banks	(B)	3
Financing Business	(FB)	4
Securities & Insurance	(SI)	4
Chemicals	(C)	2
Diversified Manufactory	(DM)	4
Iron & Steel	(IS)	2
Precision Instruments	(PI)	1
Construction	(CO)	2
Gas distribution	(GD)	2
Telecommunication	(T)	1
Office Equipments	(OE)	2

Table 1: ***Description of the data***

[6] A plus (+) or minus (-) sign may be added to the rating symbols from “AA” to “B” to indicate relative standing within each of those rating categories.

The second data set consist of daily stock return volatility for each firm of the reference entity. Daily stock prices are obtained from Tokyo Stock Exchange. Then, we estimate daily stock return volatility with GARCH (1,1). We are working with daily estimation and choose the latest observation before the trading date of the credit default swap.

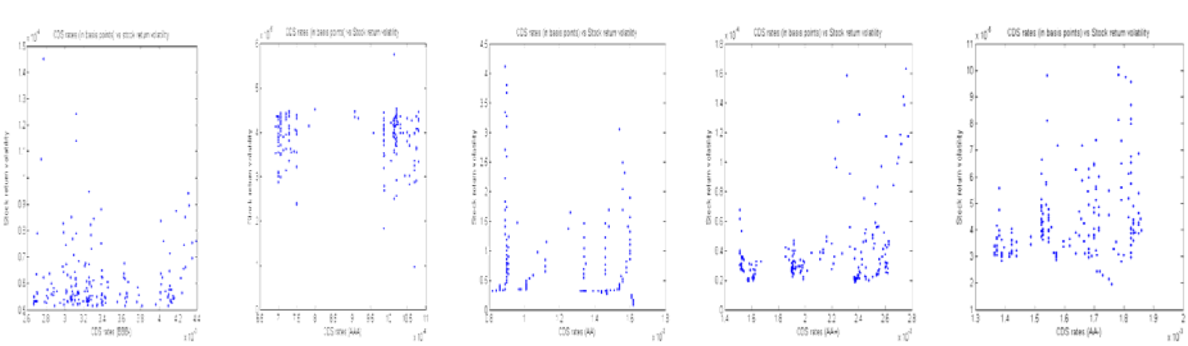
The dependence structure doesn't derive from the marginal distributions but it is totally described by the joint distribution of *uniform variates* obtained from the marginal distributions. Wang (1998) introduces dependency between risks directly on the joint distribution by approximating the cumulative distribution function via simulation.

Afterwards, we determine the corresponding copula to a bivariate distribution function from its margins. Caillaut & Guégan (2003) use the tail behaviour of the bivariate distributions of the markets to choose the copula which seems the best appropriated to characterize the dependence between Asian markets. Oakes (1994) use the inference function based on the estimation of the parameters by the maximum likelihood method. As we only use the Archimedean copula, we adopt nonparametric and semi-parametric estimation procedure. we present the generator function and the relationship with the kendall's tau and we estimate the dependence parameter θ. The appealing characteristic of Archimedean copulas is that there exists an expression linking the dependence parameter of the generator function and a measure of rank correlation (Kendall tau). According to Guégan & Ladoucette (2004), the use of Archimedean copulas are useful for empirical studies since they are easy built through Kendall's tau, also, they allow extending the modelling of bivariate series to n-variate series with $n \geq 3$ and they have closed form expressions.

4.2. Measuring dependence with Archimedean copula

4.2.1. Non-parametric estimation

A useful explanatory tool to show the possible relations between variables is the scatter plot. We plot daily credit default swap rates of different rating and the corresponding stock return volatility of reference entity.



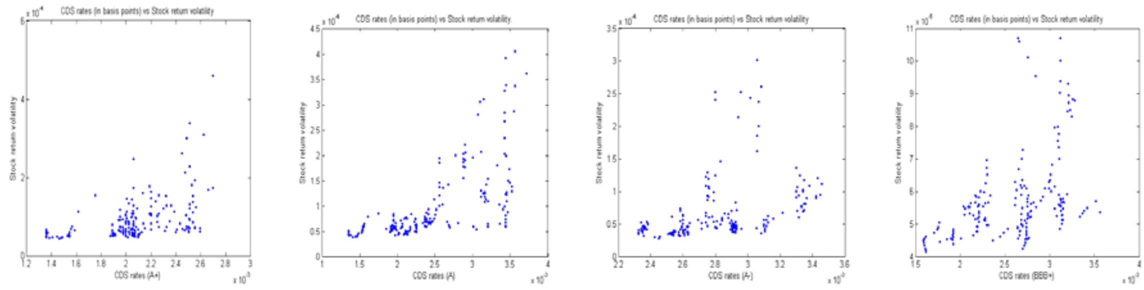


Figure 5: Scatter plot of credit default swap (CDS) rates and Stock return volatility (SRV).

According to scatter plots, we notice two facts. First, the non linear relationship between credit default swap rates and stock return volatility is clear. Second, the dependence structure behaviour changes with the rating class. Similarly, we notice that extreme values are especially important for lower rating than the higher rating class and then, the joint distributions are clearly asymmetric.

The next step consists to measure the dependence between variables. Then, the Pearson's correlation coefficient is not adapted since we have not linear relationship. It is only informative in linear and elliptical context. Serious deficiencies of Pearson correlation coefficient motivate alternative measures of dependence called rank correlation. We focus on Kendall's tau (τ) and Spearman Rho (ρ). They are nonparametric measures of dependence since they are independent of the margins[7].

The following table presents the values of Pearson correlation coefficient, kendall tau and Spearman rho under function Log transformation. We notice that, contrary to Pearson correlation coefficient, Kendall tau and Spearman rho are invariant under strictly increasing transformations of random vectors. The invariance property has practical value since the majority of financial data are non stationary time series and requires some variance-stabilizing transformation such as the log transformation. We can be assured that the transformation has not changed the copula and then, the dependence structure is preserved

Dependence measure	**CDSR × SRV**	**Log (CDSR) × Log (SRV)**
Pearson	0.659	0.563
Kendall	0,486	0,486
Spearman	0.664	0.664

Table 2: comparison between Pearson coefficient, Kendall tau and spearman rho values

Afterwards, we compute the empirical values of Kendall tau $\hat{\tau}$ and Spearman rho $\hat{\rho}$ between the different pairs of series. We report the values in the following table:

[7] See Appendix 2.

Pairs	$\hat{\tau}$	$\hat{\rho}$	**Pairs**	$\hat{\tau}$	$\hat{\rho}$
CDS**BBB**-EP	0.0646	0.0912	CDS**A**+DM	0.6479	0.8432
CDS**BBB**+F	0.1811	0.2626	CDS**AA**-F	0.1371	0.2090
CDS**BBB**+EP	0.3613	0.5074	CDS**AA**-CO	0.5592	0.7453
CDS**BBB**+LT	0.3020	0.4008	CDS**AA**-DM	0.1277	0.1894
CDS**BBB**+A	0.2864	0.3867	CDS**AA**-OE	0.1801	0.2752
CDS**BBB**+A	0.3308	0.4722	CDS**AA**-FB	0.2138	0.3161
CDS**A**-LT	0.3852	0.5680	CDS**AA**-RE	0.3158	0.4630
CDS**A**-DM	0.2969	0.4324	CDS**AA**-IS	0.0980	0.1377
CDS**A**-PI	0.0933	0.1495	CDS**AA**EP	0.4150	0.5931
CDS**A**B	0.5749	0.7760	CDS**AA**EP	0.1756	0.2648
CDS**A**FB	0.3494	0.5501	CDS**AA**OE	0.2174	0.2961
CDS**A**SI	0.4221	0.5513	CDS**AA**+DM	0.1946	0.2713
CDS**A**RE	0.1859	0.2810	CDS**AA**+SI	-0.1323	-0.1884
CDS**A**EP	0.2252	0.3560	CDS**AA**+GD	0.0172	0.0225
CDS**A**B	0.5153	0.7133	CDS**AAA**PI	-0.0510	-0.0784
CDS**A**+C	0.1600	0.2478	CDS**AAA**EP	-0.1186	-0.1948
CDS**A**+IS	-0.2539	-0.3813	CDS**AAA**SI	0.0052	0.0066
CDS**A**+DM	0.1734	0.2622	CDS**AAA**LT	0.0939	0.1386
CDS**A**+B	0.3472	0.4989	CDS**AAA**T	-7.8064e-004	-0.0104
CDS**A**+FB	0.1149	0.1727	CDS**AAA**EP	0.0625	0.1048
CDS**A**+FB	0.2244	0.3344	CDS**AAA**tokyoGD	0.0272	0.0415
CDS**A**+FB	0.2045	0.2979			

Table 3: Kendall tau $\hat{\tau}$ and Spearman rho $\hat{\rho}$ for different pairs.

We note that correlation between credit default swap rates and stock return volatility are significant, according to Kendall's tau and Spearman's Rho statistics. Also we have a positive dependence which underlines the fact that an increase in the stock return volatility generates a widening of credit default swap rates except for the highly rating class and especially AAA grade. This link depends on the economic sector and rating.

Genest & MacKay (1986) have shown a relationship between Kendall tau and the generator function of Archimedean copula. Then, we obtain the estimates of Archimedean copula parameters as presented in the following table:

Family	**Range of** θ	$\Phi(u)$	τ
Gumbel	$\theta \in [1, \infty)$	$(-Ln(u))^{\theta}$	$\frac{\theta - 1}{\theta}$
Clayton	$\theta \in [0, \infty)$	$u^{-\theta} - 1$	$\frac{\theta}{\theta + 2}$
Frank	$\theta \in (-\infty, +\infty)$	$-\ln \frac{e^{-\theta u} - 1}{e^{-\theta} - 1}$	$1 - \frac{4}{\theta}[1 - D_1(\theta)]$

Tableau 4: One parameter families of Archimedean copula

We recall that $D_1(\theta)$ is the Debye function defined as $D_n(x)=\frac{n}{x^n}\int_0^x \frac{t^n}{e^t-1}dt$ for n positive integer. Table 4 looking at the range of θ, we can note that Gumbel and Clayton copulas could only represent positive dependence structures, while Frank's copula is more flexible.

The theorem of Genest & MacKay (1986) has simplified the framework and leads us to estimate the parameters of Archimedean copula based on Kendall tau statistic. After obtaining the values of Kendall tau for each pair, we can compute the parameter α of each Archimedean copula function. The following table displays the results related to copulas parameters estimation:

Pairs	Gumbel	Clayton	Frank	Pairs	Gumbel	Clayton	Frank
CDS**BBB-**EP	1.0691	0.1381	0.5834	CDS**A**RE	1.2284	0.4567	1.7217
CDS**BBB**+F	1.2212	0.4423	1.6747	CDS**A**EP	1.2907	0.5813	2.1148
CDS**BBB**+EP	1.5657	1.1314	3.6503	CDS**A**B	2.0631	2.1263	6.0215
CDS**BBB**+LT	1.4327	0.8653	2.9401	CDS**A**+C	1.1905	0.3810	1.4707
CDS**BBB**+A	1.4013	0.8027	2.7650	CDS**A**+IS	N.S[8]	N.S	-2.4131
CDS**BBB**+A	1.4943	0.9886	3.2755	CDS**A**+DM	1.2098	0.4195	1.5999
CDS**A-**LT	1.6265	1.2531	3.9606	CDS**A**+B	1.5319	1.0637	3.4743
CDS**A-**DM	1.4223	0.8445	2.8824	CDS**A**+FB	1.1298	0.2596	1.0453
CDS**A-**PI	1.1029	0.2059	0.8460	CDS**A**+FB	1.2893	0.5786	2.1066
CDS**A**B	2.3524	2.7048	7.2951	CDS**A**+FB	1.2571	0.5141	1.9057
CDS**A**FB	1.5370	1.0741	3.5014	CDS**A**+DM	2.8401	3.6802	9.3664
CDS**A**SI	1.7304	1.4608	4.4737	CDS**AA**+DM	1.2416	0.4832	1.8074
CDS**AA-**F	1.1589	0.3178	1.2531	CDS**AA**+SI	N.S	N.S	-1.2079
CDS**AA-**CO	2.2686	2.5372	6.9309	CDS**AA**+GD	1.0175	0.0350	0.1548
CDS**AA-**DM	1.1464	0.2928	1.1647	CDS**AAA**PI	N.S	N.S	-0.4600
CDS**AA-**OE	1.2197	0.4393	1.6650	CDS**AAA**EP	N.S	N.S	-1.0797
CDS**AA-**FB	1.2719	0.5439	1.9990	CDS**AAA**SI	1.0052	0.0105	0.0468
CDS**AA-**RE	1.4616	0.9231	3.0987	CDS**AAA**LT	1.1036	0.2073	0.8512
CDS**AA-**IS	1.1086	0.2173	0.8889	CDS**AAA**T	N.S	N.S	-0.0070
CDS**AA**EP	1.7094	1.4188	4.3714	CDS**AAA**EP	1.0667	0.1333	0.5643
CDS**AA**EP	1.2130	0.4260	1.6212	CDS**AAA**tokyoGD	1.0280	0.0559	0.2449
CDS**AA**OE	1.2778	0.5556	2.0354				

Tableau 5: Nonparametric estimation of Archimedean copula parameters

We have a positive dependence between credit default swap rates and stock return volatility. After that, we address the problem of selecting the appropriate copula that better fit each pair of our data. We use the procedure developed by Genest & Rivest (1993). We adopt an unobserved random variable $Z_t = F(X_{1i}, X_{2i})$ with distribution function $K(z)=\Pr ob(Z_i \le z)$. Genest & Revest (1993) proved that the estimation of an Archimedean copula is uniquely determined by the function $K(z)=z-\frac{\Phi(z)}{\Phi^{'}(z)}$ defined on the unit interval.

To identify the generator function Φ, we adopt the following steps:

[8] Not Significant

- Estimate Kendall's correlation coefficient using the non parametric estimate.

- Construct nonparametric estimate of K by:

First, define the pseudo-observations

$Z_i = \{$**number of** (X_{1j}, X_{2j}) **such that** $X_{1j} < X_{1i}$ **and** $X_{2j} < X_{2i}\}/(n-1)$ for $i = 1,2,\ldots,n.$

Second, construct the estimate of K as $K_n(z) =$ **proportion of** Z_i $s \le z.$

- Construct a parametric estimate of K using the relationship $K_\Phi(z) = z - \frac{\Phi(z)}{\Phi'(z)}$.

For example, we use the estimate τ_n to calculate copula parameter θ, say θ_n. We use θ_n to estimate the generator function $\Phi(X)$, say $\Phi_n(X)$. Finally, we use $\Phi_n(X)$ to estimate $K_\Phi(z)$, say $K_{\Phi_n}(z)$.

We repeat the latest step for different choices of generator function (for Gumbel, Clayton and Frank) and then, we compare each parametric estimate to the nonparametric estimate constructed in step 2.

For doing this, we need the first derivative regard to z. For different Archimedean copula adopted in our study, we have:

	Generator $\Phi(z)$	**Generator first derivative** $\Phi'(z)$	**The distribution function** $K(z) = z - \frac{\Phi(z)}{\Phi'(z)}$
GUMBEL	$(-Ln(z))^\theta$	$-\theta(\ln z)^{\theta-1}\frac{1}{z}$	$z - \frac{(z \ln z)}{\theta}$
CLAYTON	$z^{-\theta} - 1$	$-\theta z.z^{-\theta-1}$	$z - \frac{(z^{\theta+1} - z)}{\theta}$
FRANK	$-\ln\frac{e^{-\theta z} - 1}{e^{-\theta} - 1}$	$\frac{\theta}{1 - e^{\theta z}}$	$z - \frac{\ln\frac{e^{-\theta z} - 1}{e^{-\theta} - 1}}{\theta}\left(e^{\theta z} - 1\right)$

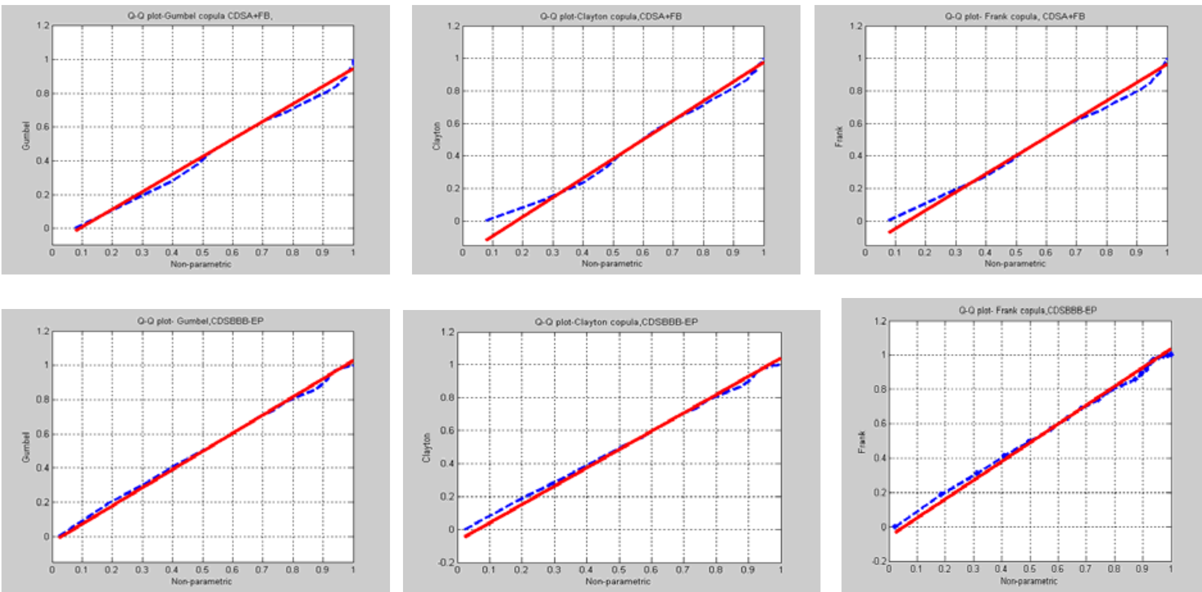
Tableau 6: Distribution function of Archimedean copula

Following Frees & Valdez (1998), the select of the Archimedean copula that fits better the data can be done by minimizing a distance such as: $\int\left[K_{\Phi_n}(z) - K_n(z)\right]^2 dK_n(z)$. We obtain the following results:

Pairs	Gumbel	Clayton	Frank	Pairs	Gumbel	Clayton	Frank
CDS**BBB**-EP	0,0034879	0,0060352	0,0046556	CDS**A**RE	0,1052822	0,1520558	0,1221179
CDS**BBB**+F	0,0290713	0,0472493	0,0339737	CDS**A**EP	0,0426521	0,0225239	0,0253067
CDS**BBB**+EP	0,014801	0,0211902	0,0143667	CDS**A**B	0,7586853	0,8027598	0,8157554
CDS**BBB**+LT	0,0503197	0,1097062	0,0544479	CDS**A**+C	0,0403233	0,0764539	0,0486567
CDS**BBB**+A	0,0272942	0,0330016	0,0288589	CDS**A**+IS	***NS***	***NS***	0,0277412
CDS**BBB**+A	0,0778352	0,0237119	0,0583566	CDS**A**+DM	0,1624071	0,1804144	0,1776127
CDS**A**-LT	0,1515985	0,2313792	0,1782139	CDS**A**+B	0,2752401	0,3330097	0,3070789
CDS**A**-DM	0,0705651	0,0596665	0,0741712	CDS**A**+FB	0,0686032	0,0901533	0,0770628
CDS**A**-PI	0,0251967	0,0181465	0,0191554	CDS**A**+FB	0,0632107	0,0897419	0,0739094
CDS**A**B	0,8835672	1,0156956	0,9530456	CDS**A**+FB	0,1257464	0,1494565	0,1349206
CDS**A**FB	0,3242598	0,4434977	0,3588152	CDS**A**+DM	0,3599568	0,4933177	0,4008972
CDS**A**SI	0,1627731	0,2339411	0,1861423	CDS**AA**+DM	0,0817801	0,1324516	0,0991841
CDS**AA**-F	0,0349088	0,0654405	0,0475897	CDS**AA**+SI	***NS***	***N.S***	0,1319259
CDS**AA**-CO	0,071043	0,1467103	0,0941097	CDS**AA**+GD	0,0153395	0,0149536	0,015257
CDS**AA**-DM	0,0146318	0,0308386	0,0209143	CDS**AAA**PI	***NS***	***N.S***	0,0062612
CDS**AA**-OE	0,2790841	0,327054	0,2823131	CDS**AAA**EP	***NS***	***N.S***	0,0197457
CDS**AA**-FB	0,0195356	0,0227957	0,0131642	CDS**AAA**SI	0,17127	0,1735721	0,1719921
CDS**AA**-RE	0,0614588	0,1115062	0,0730593	CDS**AAA**LT	0,1232786	0,158994	0,1320172
CDS**AA**-IS	0,0544375	0,0543715	0,0568632	CDS**AAA**T	***NS***	***N.S***	0,1036108
CDS**AA**EP	0,1892502	0,2527017	0,2088976	CDS**AAA**EP	0,061582	0,0610888	0,0617819
CDS**AA**EP	0,0393814	0,0440957	0,0374448	CDS**AAA**tokyoGD	0,0279227	0,0316747	0,0292264
CDS**AA**OE	0,147574	0,2225562	0,1621052				

Tableau 7: The best appropriate Archimedean copula

The quantile-quantile (Q-Q) plot is a graphical technique for determining if two data sets come from populations with a common distribution[9]. A Q-Q plot is a plot of the quantiles of the first data set against the quantiles of the second data set.



[9] A 45-degree reference line is also plotted. If the two sets come from a population with the same distribution, the points should fall approximately along this reference line.

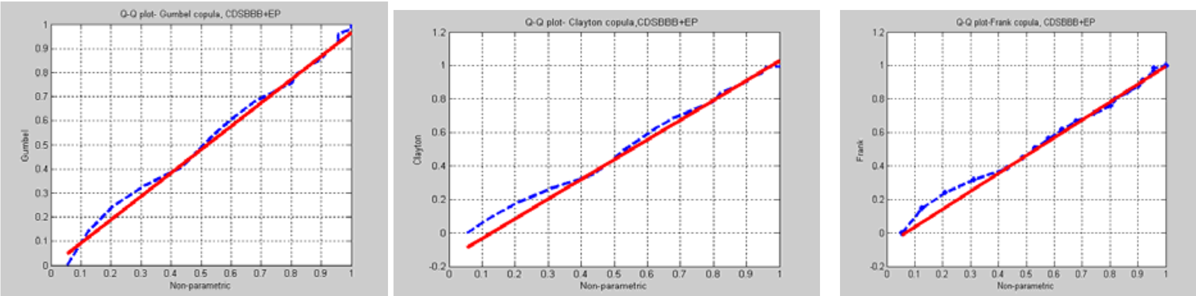


Figure 7: Q-Q plot

Analysing visually the Q-Q plot and according to our empirical result, it seems that Gumbel copula give better fits than Clayton and Frank.

4.2.2. Semi-parametric estimation

Because of the semi-parametric nature of estimation procedure, the copula parameter cannot be estimated directly by Maximum likelihood function. Genest et al (1995) propose a pseudo-likelihood procedure in which the margins are estimated by the marginal empirical distribution function. Then, the copula parameter is estimated by maximizing the pseudo-likelihood that arises when the margins are treated as known. The copula density for each Archimedean copula is presented as $\frac{\partial^2 C(u,v)}{\partial u \partial v} = -\frac{\Phi''(C(u,v))\Phi'(u)\Phi'(v)}{[\Phi'(C(u,v))]^3}$.

To examine the goodness of our estimation, we can compare the value of the negative log-likelihood functions using the Akaike information criterion, which is defined by:

AIC=2(negative log-likelihood)+2K. Where K is the number of parameters of the model in our case equals to 1. We have the same number of parameters and we estimate just one parameter $\hat{\theta}$. Then, we can simplify AIC to AIC' such as: AIC' *=2(negative log-likelihood).*

The value of AIC contains the information which estimator fits better. The best appropriate copula that best fit the data is determined by the lowest AIC value. The following table presents example of parametric estimation with Akaike information criterion:

Pairs	**Copula family**	**Likelihood** $\hat{\theta}$	AIC'
CDSAB	*Gumbel*	2,134971	-170.87
	Clayton	0,918996	-73.48
	Frank	6,047974	-136.89
CDSBBB-EP	*Gumbel*	1,09986437	-4.62
	Clayton	1,0036521	32.42
	Frank	0,60172623	-1.202

Tableau 8: Semi-Parametric with Akaike information criterion

We notice that the value of $AIC^{'}$ is the lowest for Gumbel copula. Although we have reached the same conclusion with both procedures, we got slightly different values for the dependency parameters.

5. Conclusion

Compared to the joint distribution approach or correlation based approach, a copula procedure is a more convenient tool in studying the dependence structure. When modelling with Archimedean copulas, there are procedures that could be easily applied. At a first stage, we estimate Archimedean copula parameter with nonparametric approach developed by Genest & Rivest (1993). The nonparametric estimation procedure has the advantage that we do not need to know the marginal distributions to estimate the copula parameter. At a second stage, we adopt semi-parametric estimation developed by Genest et al (1995). De- Matteis (2001) shows that neither method (nonparametric approach and parametric approach using maximum likelihood) is generally more convenient, but if there are outliers or if the marginal distributions are heavy tailed, it seems reasonable to choose the nonparametric approach. If we work with a large data set, the likelihood estimator may be more precise.

Based on the empirical findings in the paper, we suggest that the dependence structure between credit default swap rates and stock return volatility is asymmetric and positive and display right tail dependence since the Gumbel copula was confirmed by nonparametric and semi-parametric procedures. We emphasize the finding that pairs that have higher rating (AAA) present a weaker dependence coefficient and then, the impact of stock return volatility on credit default swap rates is higher for the lowest rating class.

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Appendix 1

The contour or the level curves of a copula C are given by $\{(u,v) \in \mathrm{I}^2 / C(u,v) = t\}$.

To illustrate the range of bivariate behaviour that can be represented by Archimedean copula, consider the following figures:

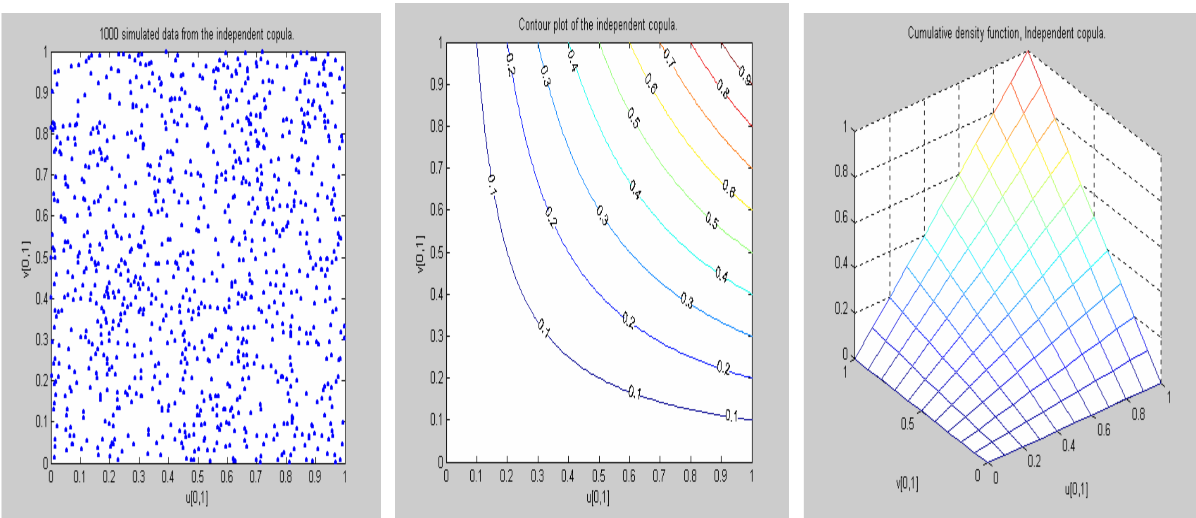

Figure 1: ***Standard uniform random variables (u,v) (Independence copula)***

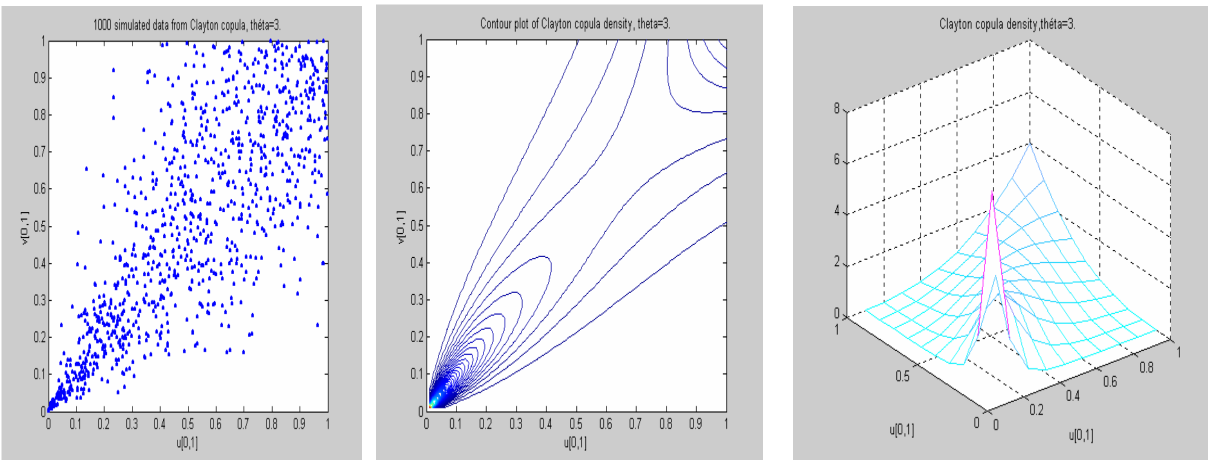

Figure 2: ***Clayton copula***

The range of bivariate behaviour that can be represented by Gumbel copula is illustrated as follow: The Clayton copula has lower tail dependence but not upper tail dependence. The contour generated by the Clayton copula implies fat tailed distribution.

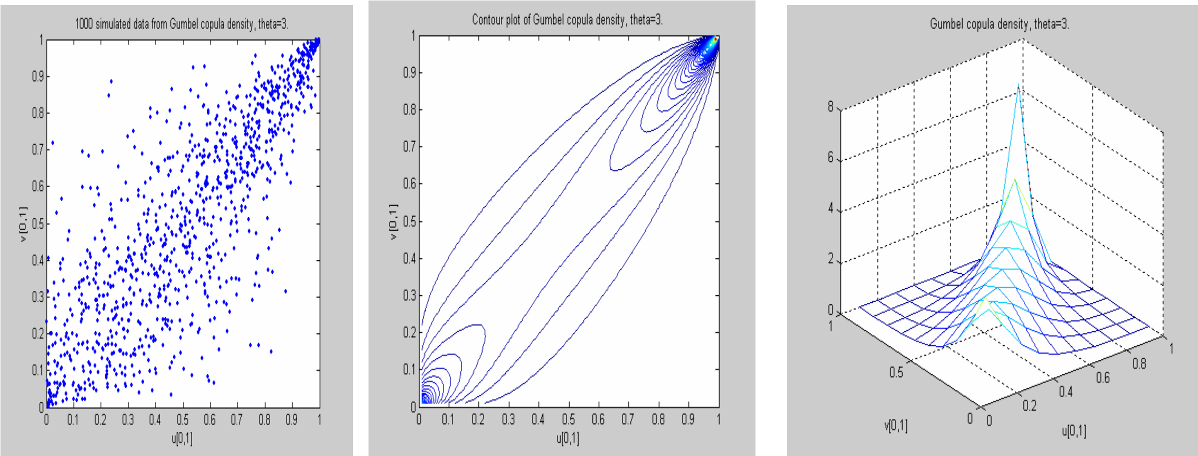

Figure 3: ***Gumbel copula***

The Gumbel copula has Upper tail dependence but not lower tail dependence. Also, this copula does not allow for negative dependence and allow for positive right tail dependence, which means the probability that both variables are in their right tails is positive.

The range of bivariate behaviour that can be represented by Frank copula is illustrated as follow:

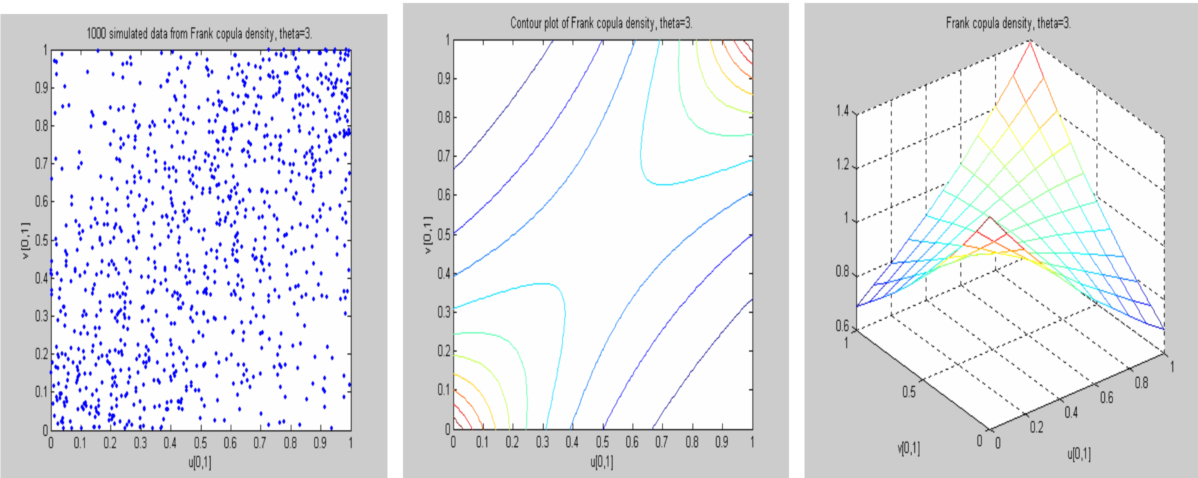

Figure 4: ***Frank copula***

The Frank copulas do not exhibit either upper or lower tail dependence. They are the only radially symmetric Archimedean copulas.

Appendix 2

Kendall's tau (τ) and Spearman Rho (ρ) can be computed based on the copula associated with the bivariate data. Schweizer & Wolff (1981) show that two standard nonparametric rank correlation can be expressed solely in terms of the copula function.

Spearman's rank correlation coefficient is a suitable measure in case of non linear relationship. Let $C(u_1, u_2) = F(F_X^{-1}(u_1), F_Y^{-1}(u_2))$ be the copula associated to the bivariate random vector X and Y. F_X and F_Y are the marginals of the distribution of X and Y random variable respectively and $(u_1, u_2) \in [0,1]^2$. The Spearman Rho (ρ) can be expressed as follows:

$$\rho = 12 \iint_{[0,1]^2} u_1 u_2 dC(u_1, u_2) - 3$$

The Kendall's tau (τ) for two random variables X and Y is the probability of concordance minus the probability of discordance. Suppose that (X,Y) and (X*,Y*) are two independent realizations of a joint distribution: $\tau = 4 \iint_{[0,1]^2} C(u_1, u_2) dC(u_1, u_2) - 1$

$$= \mathbf{Prob}\left\{\left(X^* - X\right)\left(Y^* - Y\right) \geq 0\right\} - \mathbf{Prob}\left\{\left(X^* - X\right)\left(Y^* - Y\right) < 0\right\}$$

For simplicity, it is assumed that the marginal distributions are continuous. Following Genest & MacKay (1986), Kendall tau verifies the following properties:

- $-1 \leq \tau \leq 1$
- τ is invariant under monotone transformations: if f and g are monotone increasing or decreasing functions, then $\tau(f(X), g(Y)) = \tau(X, Y)$
- $\tau = 0$ if X and Y are independent (but not conversely).