TAKEN TO THE LIMIT:
SIMPLE AND NOT-SO-SIMPLE LOAN LOSS DISTRIBUTIONS

PHILIPP J. SCHÖNBUCHER

Department of Statistics, Bonn University

April 2002
This version: August 2002

Abstract. Formulae for the distribution of the losses of a loan portfolio that are both realistic and simple enough to be implemented in a spreadsheet are hard to come by. The most prominent example is the Vasicek (1987) formula which is based upon a simplified version of the multivariate Merton (1974) model. Using an algorithm from the theory of Archimedean Copula functions, this paper gives some more limiting loss distributions which are driven by random variables with different dependency structures.

1. Introduction

In an influential paper, Vasicek (1987, 1997) showed that in a simplified multi-obligor version of the Merton (1974) credit risk model, the distribution of the losses of a large loan portfolio can be described by the inverse Gaussian distribution function. In his setup, the probability that the fraction L of defaults in the portfolio is less than a given level q is given by

$$\mathbf{P}\left[\, L \leq q \,\right] = \Phi\left(\frac{1}{\sqrt{\varrho}} \left(\sqrt{1-\varrho}\; \Phi^{-1}(q) - \Phi^{-1}(p) \right) \right). \tag{1}$$

where p is the default probability of any individual obligor in the portfolio, and ϱ is the asset value correlation between any two obligors ($\Phi(\cdot)$ denotes the cumulative standard normal distribution function).

Usually the loss distribution of a credit risk model can only be determined using lengthy numerical simulations, thus a simple closed-form solution like (1) which

---

JEL Classification. G 13.

Key words and phrases. Copula functions, credit risk, credit portfolio models.

Financial assistance by the DFG is gratefully acknowledged.

Comments and suggestions are welcome, all errors are my own.

involves just two parameters has a lot of appeal: It can be very useful to understand the behaviour of the more complex variants of the model, to find benchmark parameter values that can be fitted to historical observations, or simply as a "quick and dirty" first approximation in all situations when setting up a full credit risk model would take too long. Furthermore, as the Credit Metrics model (Gupton et al. (1997)) is also based upon the Merton (1974) firm's value setup, Vasicek's result can also be regarded as a limiting case of this very popular model, and many of the qualitative features of the Credit Metrics model can be analysed in closed-form without having to resort to the usual lengthy simulations. The accuracy of the approximation is remarkably good, the approximation error becomes unacceptable only when very low asset value correlations $\varrho < 1\%$, very few obligors (¡20) or extremely heterogeneous exposure sizes (e.g. one dominating obligor) are considered.

Thus it is no surprise that the Vasicek model has been quickly adopted in practice, large portfolios are managed on the basis of (1) and the relationship is also used in a regulatory context to set risk measures for credit exposures.

Despite its widespread use, the Vasicek model does have some shortcomings beyond the obvious over-simplification of identical default risks and exposure sizes of the obligors. For instance, there are significant difficulties replicating the *qualitative* shape of the loss distribution. For a given default probability parameter p, one can only vary ϱ to fit both the main part of the loss distribution and the tail of the distribution. If a manager of a CDO wants to calibrate ϱ in such a way that tails of the distribution are fitted well (i.e. such that the senior and super-senior tranche of his CDO are priced correctly), then he may experience serious mispricing of the main body of the distribution (i.e. the mezzanine and equity tranches). Furthermore, the shape of the distribution changes significantly when different time horizons (and thus default probabilities) are considered. Both problems have their roots in the implicit assumption of a joint Gaussian distribution of the obligors' asset value processes which imply a very specific transition from the limiting case of independence ($\varrho = 0$, all probability at $L = p$) to the fully dependent case ($\varrho = 100\%$, all probability at $L = 0$ and $L = 100\%$).

In this paper we will give a class of similar approximative loss distributions of large portfolios where we do not use multivariate normally distributed random variables to trigger defaults. Instead we model the dependencies between the defaults using *Archimedean copula functions.* Archimedean copula functions are a tractable class of joint distribution functions with characteristics that can be significantly different from the characteristics of a multivariate normal distribution function.

The aim of this exercise is to analyse the effects that the specification of the dependency structure between the individual default events can have on the loss distribution of the whole portfolio. Therefore we can use a very stylised model and we can keep all other parameters fixed, such as individual default probabilities, exposure size, and even the pairwise default correlations, in order to isolate the effects of the dependency structure.

The main tool that we use to achieve this result is an algorithm for the generation of random variates with a given Archimedean copula function as joint distribution function which was first proposed by Marshall and Olkin (1988). This algorithm in itself may already be a valuable tool for the reader, because a major obstacle against the wider adoption of alternative dependency models in risk management to date was the lack of efficient numerical implementation schemes for large-scale simulations.

In the next section we will introduce the notion of a copula function and motivate why it is a good idea to check the results of a Gaussian model against alternative specifications of the dependency structure. Then we will introduce the special class of Archimedean copulae and give the algorithm to generate random variates whose dependency structure is described by an Archimedean copula. We specify the algorithm for the Clayton, the Gumbel and the Frank copulae, three special cases where the data generating process can be given in closed-form. Then, we enter the portfolio credit risk model and derive closed-form formulae for the loss distribution of a large loan portfolio whose defaults are driven by random variables with this type of dependence. The numerical implementation and comparison of these loss distributions will show that the nature of the dependency structure does have a significant effect on the loss distribution, even if the default correlation between any two obligors is held fixed.

The literature on dependency modelling and copula functions has grown substantially in recent years, therefore we cannot give a full overview here. The reader is referred to the excellent (but slightly technical) textbook by Joe (1997) for the basics, another popular textbook is Nelsen (1999). The standard reference for the generation of non-uniform random variates is Devroye (1986) which contains hundreds of algorithms. For the application of copula functions to credit risk modelling we should mention Li (2000) and Schönbucher and Schubert (2001) and Frey and McNeil (2001). Frey and McNeil (2001) give an overview over the different approaches to portfolio credit risk modelling with a particular focus on the dependency structures implied by the models. They also analyse the loss distribution of large loan portfolios in the more general setup of Bernoulli Mixture models, and compare the differences between a Gaussian (e.g. Credit Metrics) and t-Copula dependency structure.

2. Copula Functions and Laplace Transforms

2.1. Copula Functions.

Whenever several dependent dimensions of uncertainty have to be modelled, the standard (and often also the only) approach is to somehow transform the problem in such a way that a multivariate normal distribution can be used to model the uncertainty. In the modelling of equities, exchange rates and interest-rates, multivariate lognormal distributions are used (i.e. exponentials of normals), squared Gaussian and related models are also popular, and in portfolio credit risk modelling the Credit Metrics model is driven by a multivariate normal distribution of the obligor's asset value processes.

Using a multivariate normal distribution as driver of the model leads to a so-called *Gaussian dependency structure* of the key variables of the resulting model. This can be a restrictive modelling choice, it is just one out of an infinite number of possible joint distribution functions. The full set of all possible dependencies between I random variables is given by the set of all I-dimensional copula functions.

So what is a Copula? Roughly speaking:

An I-dimensional Copula is a distribution function on $[0,1]^I$ with uniform marginal distributions.

That is all. Copulas concentrate on the dependency, so the marginal distribution is irrelevant. It is set to a uniform distribution because this makes the later incorporation of other marginal distributions straightforward, and we recover the benchmark case of the uniform distribution on $[0,1]$ if we ignore the other $I-1$ random variables.

The technical definitions of copulas that are given in the mathematical literature often look quite different, but to a financial modeller, this is the definition to build an intuition from. The reason why Copulae provide a useful framework to analyse dependencies between random variables is the fact that to every multivariate distribution function there is a Copula which contains all information on dependence. This is the essence of the following theorem by Sklar:

Theorem 2.1 (Sklar)**.**
$X_1, \dots, X_I$ with marginal distribution functions $F_1, F_2, \dots, F_I$ and joint distribution function F. Then there exists a I dimensional copula C such that

$$\begin{aligned} F(x_1, \dots, x_I) &= C(F_1(x_1), F_2(x_2), \dots, F_N(x_I)) \qquad \forall \mathbf{x} \in \mathbb{R}^I, \\ C(u_1, \dots, u_I) &= F(F_1^{[-1]}(u_1), \dots, F_1^{[-1]}(u_I)). \end{aligned}$$

If $F_1, F_2, \dots, F_N$ are continuous, then C is unique.
In particular, the copula $C(\cdot)$ is the distribution function of the transformed random variables $U_1 = F_1(X_1), \dots, U_I = F_I(X_I)$.

So, to every distribution function on $\mathbb{R}^I$, there is a corresponding copula function. For example, if the random variables X_i are independent, then the *independence copula* is just the product of the u_i

$$C(u_1, \dots, u_I) = u_1 \cdot u_2 \cdot \ldots \cdot u_I.$$

If $X_1, \dots, X_I$ have a multivariate normal distribution with covariance matrix Σ and mean zero (for simplicity), then the *Gaussian copula* is reached

$$C(x_1, \dots, x_I) = \Phi_{\Sigma,0}(\Phi_{\sigma_{11}^2}^{[-1]}(x_1), \dots, \Phi_{\sigma_{II}^2}^{[-1]}(x_I)),$$

where $\Phi_{\sigma^2}()$ is the univariate cumulative normal distribution function with variance σ^2 and mean zero, and Φ_Σ the multivariate cumulative normal distribution function with covariance matrix Σ.

As the next section will show, there are even more possibilities.

2.2. **Archimedean Copula Functions.** Copula functions do not impose any restrictions on the model at all, so in order to reach a model that is to be useful in practical applications, a particular specification of the copula must be chosen. As we want to provide an alternative to the Gaussian model, we use the Archimedean copula functions as a benchmark model.

Definition 2.2 (Archimedean Copula)**.**

(i) *An* Archimedean copula *function* $C : [0,1]^I \to [0,1]$ *is a copula function which can be represented in the following form*

$$C(\mathbf{x}) = \phi^{[-1]}\Big(\sum_{i=1}^{I} \phi(x_i)\Big), \tag{2}$$

with a suitable function $\phi : [0,1] \to \mathbb{R}_+$ *with* $\phi(1) = 0$*,* $\phi(0) = \infty$*.*

(ii) *The function* $\phi : [0,1] \to \mathbb{R}_+$ *is called the* generator *of the copula.*

Not every function ϕ is a suitable generator for a copula function, there are restrictions on the signs of the derivatives of ϕ which become more stringent with increasing dimension I. But in the following case the existence of the copula can be ensured:

If $F(x)$ is a distribution function of a positive random variable with $F(x = 0) = 0$ and

$$\hat{F}(y) = \int_0^\infty e^{-yx} dF(x)$$

is its Laplace transform, then $\phi(t) := \hat{F}^{[-1]}(t)$ is the generator of a Archimedean copula of dimension I for every $I > 0$. (In fact, $\phi^{[-1]}()$ must be a Laplace transform if it is to be an admissible generator for *any* dimension $I > 0$.)

From equation (2) we can see that Archimedean copula models are *exchangeable*, i.e. the dependency between any two (or i) different risk factors does not depend on the question *which* two (or i) risk factors were chosen. For our aim of assessing portfolio credit risk in large, homogeneous portfolios this does not pose a major restriction, in fact it is a desirable property. (For other applications this may not be the case.)

In table 2.2 we give some popular specifications of the generator functions ϕ and their inverses $\phi^{[-1]}$, together with the inverse Laplace transform of the inverse generator $\psi(s) = \mathcal{L}^{[-1]}_{\phi^{[-1]}}(s)$. We will need Laplace transform and inverse Laplace transforms later on, so this may be a good place to define them:

Definition 2.3 (Laplace Transform)**.**
Let Y be a nonnegative random variable with distribution function $G(y)$ and density function $g(y)$ (if a density exists). Then

(i) *The Laplace transform of Y is defined as*

$$\mathcal{L}_Y(t) := \mathbf{E}\left[\, e^{-tY} \,\right] = \int_0^\infty e^{-ty} dG(y) = \int_0^\infty e^{-ty} g(y) dy =: \mathcal{L}_g(t), \qquad \forall t \geq 0. \tag{3}$$

(ii) *Let* $\psi : \mathbb{R}_+ \to [0,1]$*. If a solution exists, the inverse Laplace transform* $\mathcal{L}_\psi^{[-1]}$ *of* ψ *is defined as the function* $\chi : \mathbb{R}_+ \to [0,1]$ *which solves*

$$\mathcal{L}_\chi(t) = \int_0^\infty e^{-ty}\chi(y)dy = \psi(t), \qquad \forall t \geq 0.$$

(iii) *The distribution of* Y *is uniquely characterised by its Laplace transform.*

1.	Name:	Clayton
	$\phi(t) =$	$(t^{-\theta} - 1)$
	$\phi^{[-1]}(s) =$	$(1 + s)^{-1/\theta}$
	Parameter:	$\theta \geq 0$
	Y-Distribution:	Gamma $(1/\theta)$
	Density of Y:	$\frac{1}{\Gamma(1/\theta)} e^{-y} y^{(1-\theta)/\theta}$
2.	Name:	Gumbel
	$\phi(t) =$	$(-\ln t)^{\theta}$
	$\phi^{[-1]}(t) =$	$e^{(-s^{1/\theta})}$
	Parameter:	$\theta \geq 1$
	Y-Distribution:	α-stable, $\alpha = 1/\theta$
	Density of Y:	(no closed-form is known)
3.	Name:	Frank
	$\phi(t) =$	$-\ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}$
	$\phi^{[-1]}(t) =$	$-\frac{1}{\theta} \ln[1 - e^{-s}(1 - e^{-\theta})]$
	Parameter:	$\theta \in \mathbb{R} \backslash \{0\}$
	Y-Distribution:	Logarithmic series on $\mathbb{N}_+$ with $\alpha = (1 - e^{-\theta})$
	Distribution of Y:	$\mathbf{P}\left[\, Y = k \,\right] = \frac{-1}{\ln(1-\alpha)} \frac{\alpha^k}{k}$

TABLE 1. Some generators for Archimedean copulas, their inverses and their Laplace transforms. Source: Marshall and Olkin (1988).

3. Generation of Copula-Dependent Random Numbers

Despite the importance of an accurate model for the dependency structure of the returns of the assets in a portfolio, an obstacle for practical implementation of any copula-based model was the absence of an efficient method for generating Copula-dependent random variates. These dependent random variates are essential for the simulation of the portfolio's risk/return profile, and also for the development and testing of estimation methods for the parameters of these distributions.

The most frequently used method the conditional distributions method which involves a differentiation step for each dimension of the problem. For this reason it is not practical in dimensions larger than ten. As an alternative we propose a method which is based upon the representation of a large class of Copula functions with Laplace transforms and mixtures of powers as described in Joe (1997).

Our strategy for the sampling of a random vector X with the distribution function above is the following algorithm by Marshall and Olkin (1988).

Proposition 3.1 (Marshall / Olkin 1988)**.**
Let $\phi^{[-1]} : \mathbb{R}_+ \to [0,1]$ *and* $\phi : [0,1] \to \mathbb{R}_+$ *be continuous, strictly decreasing functions. Follow the following algorithm*

(a) Draw $U_1, \dots, U_I$ *i.i.d. uniformly distributed on* $[0,1]$*.*

(b) Draw the mixing variable Y *with the following properties:*

- *We call* Y*'s distribution function* G *(and its density* g *if a density exists).*
- Y *is independent of* $U_1, \dots, U_I$
- Y*'s Laplace transform is* $\phi^{[-1]}(\cdot)$

$$\mathcal{L}_G(s) := \mathbf{E}\left[\, e^{-sY} \,\right] = \int_0^\infty e^{-sy} dG(y) = \phi^{[-1]}(s). \tag{4}$$

(c) Define

$$X_i := \phi^{[-1]}(-\frac{1}{Y}\ln U_i) \qquad 1 \le i \le I. \tag{5}$$

Then the joint distribution function of the X_i*,* $1 \le i \le I$ *is*

$$\mathbf{P}\left[\, \mathbf{X} \le \mathbf{x} \,\right] = \phi^{[-1]}\left(\sum_{i=1}^{I} \phi(x_i) \right),$$

the X_i *have the Archimedean Copula function with generator* $\phi(\cdot)$ *as distribution function.*

From (4) follows that the density of Y is the inverse Laplace transform of $\phi^{[-1]}$. In many cases the distribution of Y can already be identified by looking at $\mathcal{L}^{[-1]}(\phi^{[-1]})$ and an efficient simulation algorithm may already be available. Otherwise Y can also be generated using a uniform random variable V as follows

$$Y := G^{[-1]}(V) \qquad \text{where} \quad G = \mathcal{L}^{[-1]}(\phi^{[-1]}).$$

Proof. First note that

$$\mathbf{P}\left[\, X_i \leq x_i \mid Y \,\right] = \exp\{-\phi(x_i) Y\},$$

and that the unconditional distribution function of Y is G. The claim of the proposition follows by using iterated expectations:

$$\begin{aligned} \text{(6)} \quad \mathbf{P}\left[\, \mathbf{X} \leq \mathbf{x} \,\right] &= \mathbf{E}\left[\prod_{i=1}^{I} \mathbf{P}\left[\, X_i \leq x_i \mid Y \,\right] \right] = \mathbf{E}\left[\prod_{i=1}^{I} \exp\{-\phi(x_i) Y\} \right] \\ &= \mathbf{E}\left[\exp\{-Y \sum_{i=1}^{I} \phi(x_i)\} \right] = \mathcal{L}_G(\sum_{i=1}^{I} \phi(x_i)) = \phi^{[-1]}(\sum_{i=1}^{I} \phi(x_i)). \end{aligned}$$

□

The key point about the algorithm shown above is that *conditional on the realisation of Y, the random variables X_i are independent.* This conditional independence property was exploited in the proof of the algorithm, and it will also drive the results in the credit risk model.

4. THE PORTFOLIO CREDIT RISK MODEL

We now have a recipe (a set of recipes) to generate a set of I dependent random variables with uniform marginal distributions. Let us use this recipe to define a simple portfolio default risk model. The model setup is as follows:

Assumption 1 (Finite Portfolio)**.**

- *There are I obligors, we consider defaults up to a fixed time-horizon T.*
- *All obligors have the same exposure size and the same loss in default. Thus, the number D of defaults is sufficient to determine the loss of the portfolio.*
- *Obligor i has the default probability p_i until T.*
- *Obligor i defaults, if and only if $X_i \leq p_i$, where X_i is generated by the algorithm of proposition 3.1.*

4.1. **The Loan Loss Distribution for a Finite Portfolio.** In this setup, the loan loss distribution can be easily derived by conditioning on the mixing variable Y. *Conditional* on $Y = y$, the default probability of an obligor i is:

$$\begin{aligned} p_i(y) &:= \mathbf{P}\,[\, X_i \le p_i \,|\, Y = y \,] = \mathbf{P}\left[\, \phi^{[-1]}(-\frac{1}{y}\ln U_i) \le p_i \,\right] \\ &= \mathbf{P}\left[\, -\frac{1}{y}\ln U_i \ge \phi(p_i) \,\right] = \mathbf{P}\,[\, \ln U_i \le -y\phi(p_i) \,] \\ &= \mathbf{P}\,[\, U_i \le \exp\{-y\phi(p_i)\} \,] = \exp\{-y\phi(p_i)\} \end{aligned}$$

If all obligors have the same unconditional default probability $p = p_i, \quad \forall i \le I$, then $p(y) = \exp\{-y\phi(p)\}$ and the probability of k defaults in the portfolio is

$$\mathbf{P}\,[\, D = k \,] = \int_0^\infty \binom{I}{k} p^k(y)(1-p(y))^{I-k} G(dy). \tag{7}$$

4.2. **The Large Portfolio Approximation.**

Assumption 2 (Large Portfolio).
In addition to assumption 1 we assume

- *All obligors have the same unconditional default probability p.*
- *The number of obligors I is very large ($I \to \infty$), the relevant quantity for the portfolio risk is the* fraction L of defaulted obligors *in the portfolio.*

By the law of large numbers, the fraction L of defaults will almost surely be $p(y)$ in the limit of the very large portfolio, whenever the mixing variable Y has taken the value of y. Thus, the probability of having more than a fraction q of defaults in the portfolio is

$$\begin{aligned} \mathbf{P}\,[\, L \le q \,] &= \mathbf{P}\,[\, p(Y) \le q \,] = \mathbf{P}\,[\, \exp\{-Y\phi(p)\} \le q \,] = \mathbf{P}\,[\, -Y\phi(p) \le \ln q \,] \\ &= \mathbf{P}\left[\, -Y \le \frac{\ln q}{\phi(p)} \,\right] = \mathbf{P}\left[\, Y \ge -\frac{\ln q}{\phi(p)} \,\right] = 1 - G(-\frac{\ln q}{\phi(p)}). \end{aligned}$$

The distribution F and the density f of the limiting loss distribution are thus

$$F(q) = 1 - G(-\frac{\ln q}{\phi(p)}), \tag{8}$$

$$f(q) = \frac{1}{q\phi(p)} g(-\frac{\ln q}{\phi(p)}), \tag{9}$$

where we assumed in (9) that the mixing variable has a density.

5. SOME EXAMPLES

Armed with the large portfolio loss distribution functions (8) and (9), we can now analyse the effects of using different dependency specifications (i.e. different copulae, generated by different generator functions ϕ) and compare them to the standard Gaussian specification used by Vasicek.

We only compare the Clayton and the Gumbel model to the Gaussian case, and excluded the Frank copula. This was done because the mixing variable in the Frank copula does not have a density (it is integer-valued), and because the main effects should already become clear with these two comparison cases.

We use the following case as our benchmark:

Assumption 3 (Benchmark Case)**.**

- *The individual default probability is* $p = 5\%$.
- *The linear correlation between two default events is* $\rho = 10\%$.

5.1. **The Vasicek Setup.** First, we define our benchmark case, the Gaussian model used by Vasicek.

Assumption 4 (Vasicek-Model)**.**

(1) *The default of each obligor* i *is triggered by the realisation of the value* V_i *of the assets of its firm.*

(2) V_i *is normally distributed. Without loss of generality*[1], *the* V_i *are standardised* $V_i \sim \Phi(0,1)$.

(3) *Obligor* i *defaults if its firm's value* V_i *is below a barrier* K, *i.e. if* $V_i \leq K$. K *is chosen such that the individual default probability* p *is matched:* $p = \Phi(K)$.

(4) *The values of the assets of the obligors are driven by:* one common factor Y, *and an idiosyncratic standard normal noise component* ϵ_i

$$V_i(T) = \sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, \epsilon_i \qquad \forall\, i \leq I,$$

where Y *and* ϵ_i, $i \leq I$ *are i.i.d.* $\Phi(0,1)$*-distributed.*

Again, conditional on the realisation of the systematic factor Y, the firm's values and the defaults are *independent*, only now the default risk enters additively as a

---

[1] This amounts to a shift in the coordinate system and a subsequent linear scaling. As the default barrier K will be chosen to fit the default probability (and thus implicitly follows the same transformation), this transformation does not change the structure of the model.

systematic factor Y in the evolution of the firm's asset values. The *asset correlation* between two obligors is $\varrho = \mathbf{E}\,[\,V_i(T)\,V_j(T)\,]$. This model is very similar to the JPMorgan Credit Metrics model, it can be transformed to our Copula-setup by defining

$$X_i := \Phi(V_i) \qquad \text{and} \qquad p := \Phi(K).$$

Thus, the Vasicek model can be written as a Copula model with a Gaussian Copula function. The resulting loss distribution in the large portfolio setup is

$$F(q) := \mathbf{P}\,[\,L \leq q\,] = \Phi\left(\frac{1}{\sqrt{\varrho}}\left(\sqrt{1-\varrho}\,\Phi^{-1}(q) - \Phi^{-1}(p)\right)\right).$$

The probability density function $f(q)$:

$$f(q) = \sqrt{\frac{1-\varrho}{\varrho}}\exp\left\{\frac{1}{2}(\Phi^{-1}(q))^2 - \frac{1}{2\varrho}\left(\Phi^{-1}(p) - \sqrt{1-\varrho}\,\Phi^{-1}(q)\right)^2\right\}.$$

Figure 1 shows the limiting large portfolio loss distribution for various values of the asset correlation. In order to be able to distinguish the tail behaviour, the figure is also shown in log-scale. For relatively small values of the asset correlation, the loss distribution is peaked around the average default rate 5%, but with increasing asset value correlation (and thus default correlation), the distribution skewed and has a single (but finite) peak at zero. This is a "balancing" effect that compensates the shift of probability mass into the tail of the loss distribution. Figure 2 shows the development of the loss distribution if the asset correlation is increased to extremely large levels. In extreme cases, the distribution exhibits a second peak at $q = 1$, i.e. 100% losses. The loss distribution approaches the scenario with the highest default dependency, this is the scenario in which either all firms default (with 5% probability), or none (with 95% probability).

5.2. **The Clayton Copula.** The density of the loss distribution in the Clayton copula model is given by (9), where $\phi()$ is the generator of the Clayton copula and $g()$ the density of a Gamma distribution with parameter $1/\theta$.

Figures 3 and 4 show the loan loss distributions in the Clayton model for small ($\theta = 5\%, 10\%, 18.12\%$) and large ($\theta = 50\%, 100\%, 200\%$) parameter values. (The parameter θ in the Clayton model is unbounded.) The qualitative behaviour of the model resembles that of the Gaussian/Vasicek model: Again we have a single peak around the average loss rate in the cases of low default dependency, the distribution shifts to skewed distribution with a single peak at $q = 0$ as the dependency is

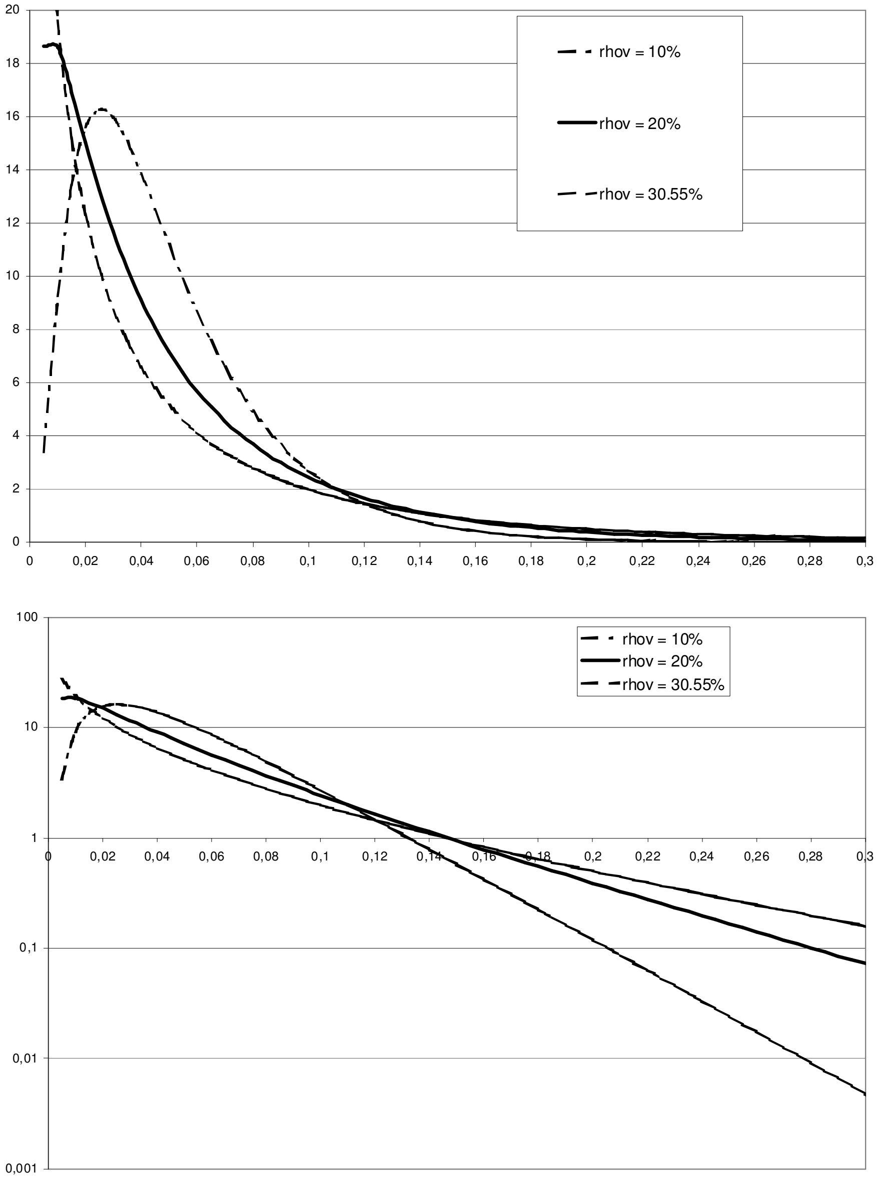

FIGURE 1. Loan loss distributions for the Vasicek model, $p = 5\%$, various asset correlations ($\varrho_V = 10\%, 20\%, 30,55\%$).

increased, and finally, for very large dependencies, we approach the extreme case with a second peak at 100% defaults.

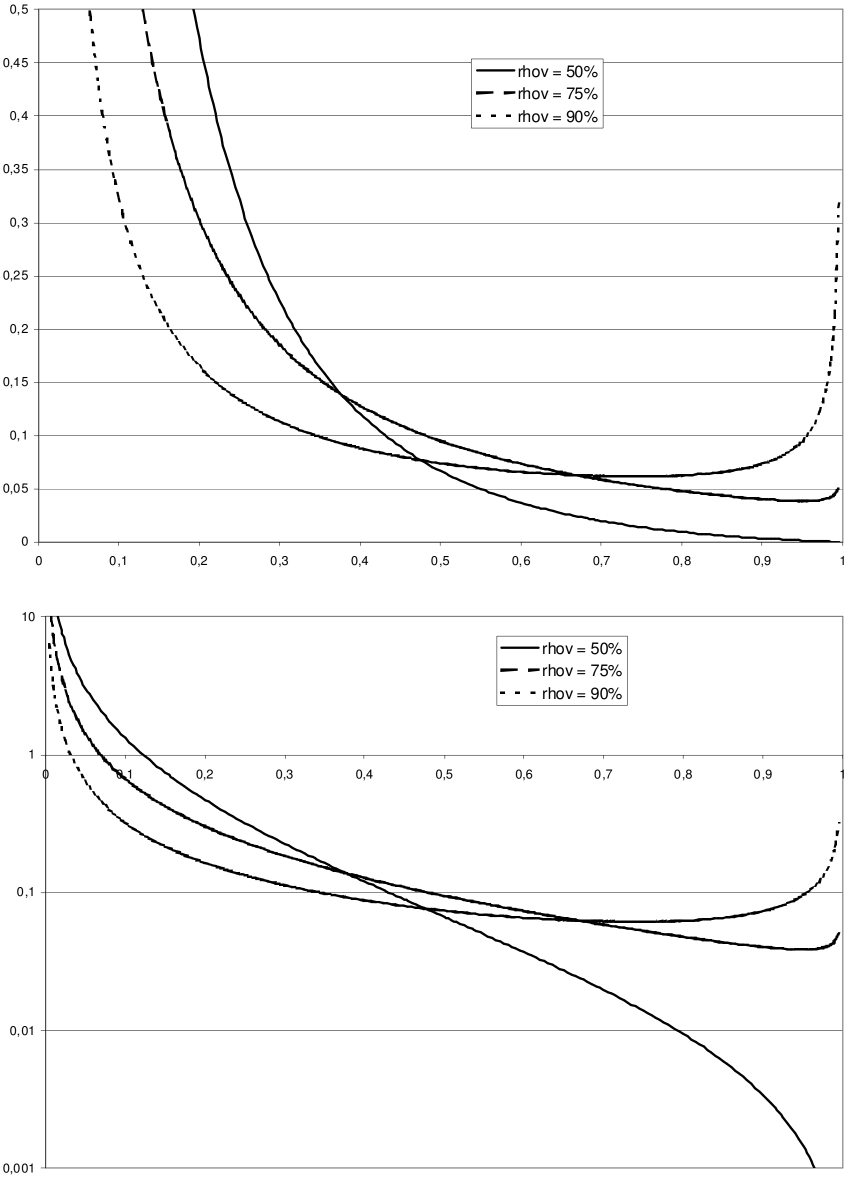

FIGURE 2. Loan loss distributions for the Vasicek model, $p = 5\%$, various large asset correlations ($\varrho_V = 50\%, 75\%, 90\%$).

5.3. **The Gumbel Copula.** For the Gumbel copula we again must substitute in (9), but now $\phi()$ is the generator of the Gumbel copula and $g()$ the density[2] of a α-stable distribution with parameter $\alpha = 1/\theta$.

Figures 5 and 6 show the loan loss distributions in the Gumbel model. We see a markedly different behaviour from the Gaussian and the Clayton case. This

---

[2] There is no closed-form solution for the density of a α-stable distribution, but it can be readily evaluated from its Fourier transform.

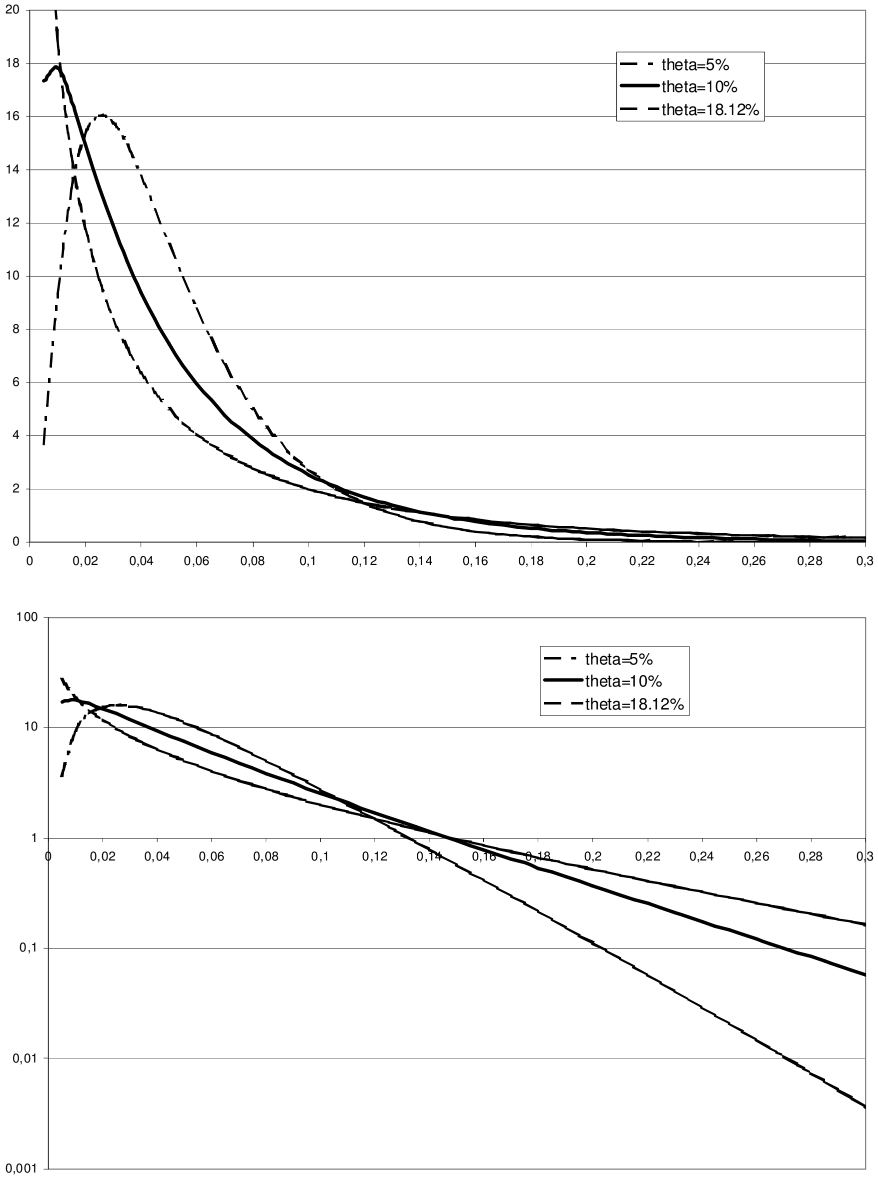

Figure 3. Loan loss distributions for the Clayton model, $p = 5\%$, parameter ($\theta = 5\%, 10\%, 18.12\%$). ($\theta = 0$ corresponds to independence, dependence increases with θ.

behaviour is driven by the special properties of the Gumbel Copula: The Gumbel copula exhibits strong *upper and lower tail dependency.* This means, that – even if linear correlations or similar measures look the same – the Gumbel distribution implies far more joint extreme events than for example the Gaussian copula which has no tail dependency. (The Clayton copula exhibits lower tail dependency, i.e. extreme movements only cluster in one direction.)

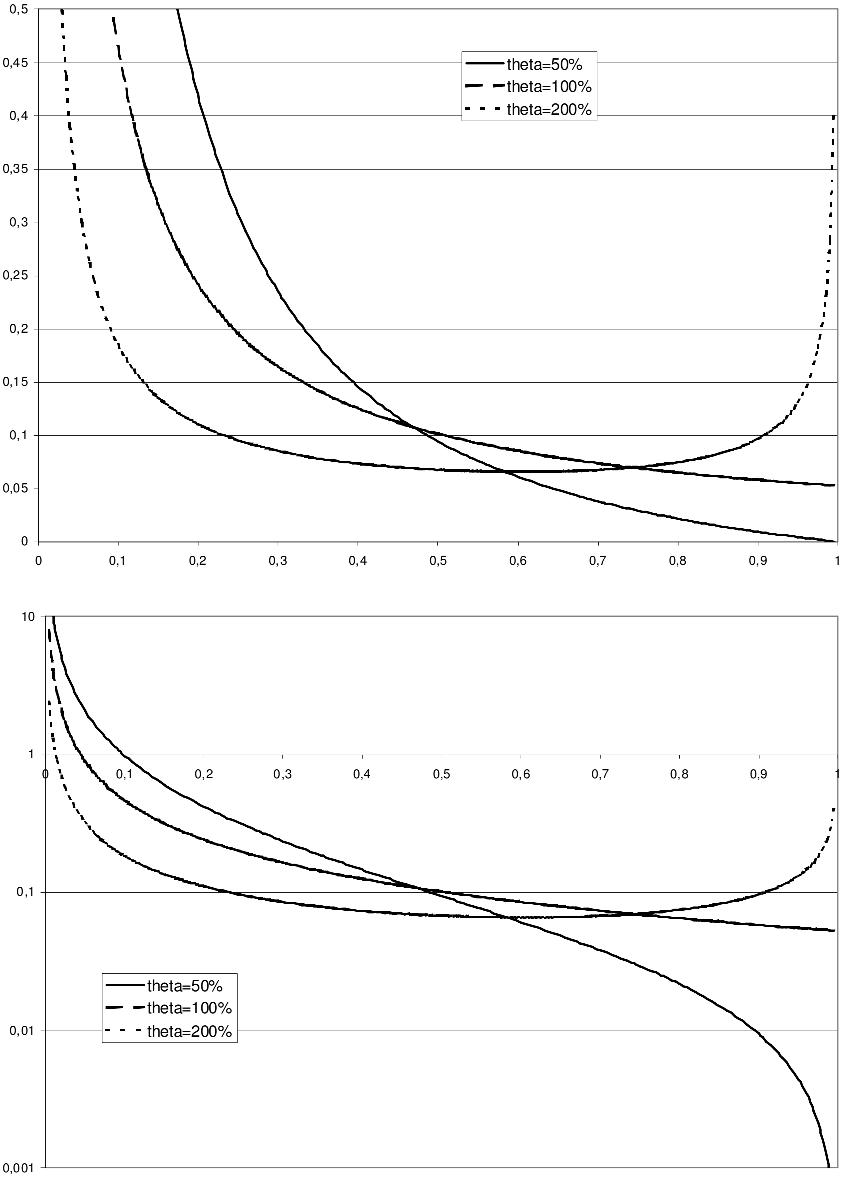

Figure 4. Loan loss distributions for the Clayton model, $p = 5\%$, large dependency parameter ($\theta = 50\%, 100\%, 200\%$).

Figure 5 shows very clearly the implications of the strong tail dependency: As we move away from the independence case ($\theta = 1$, all mass at $q = p = 5\%$), the probability mass is not simply flattening out and widening a bit as it did in the Gaussian case or the Clayton copula. No: The probability mass moves directly to the extreme event, here the "no defaults at all" ($q = 0$) event. Therefore, a second peak appears at $q = 0$ with a trough of probability for the intermediate events, in fact, at $q = 0$ there is a singularity in the density of the loss distribution. This

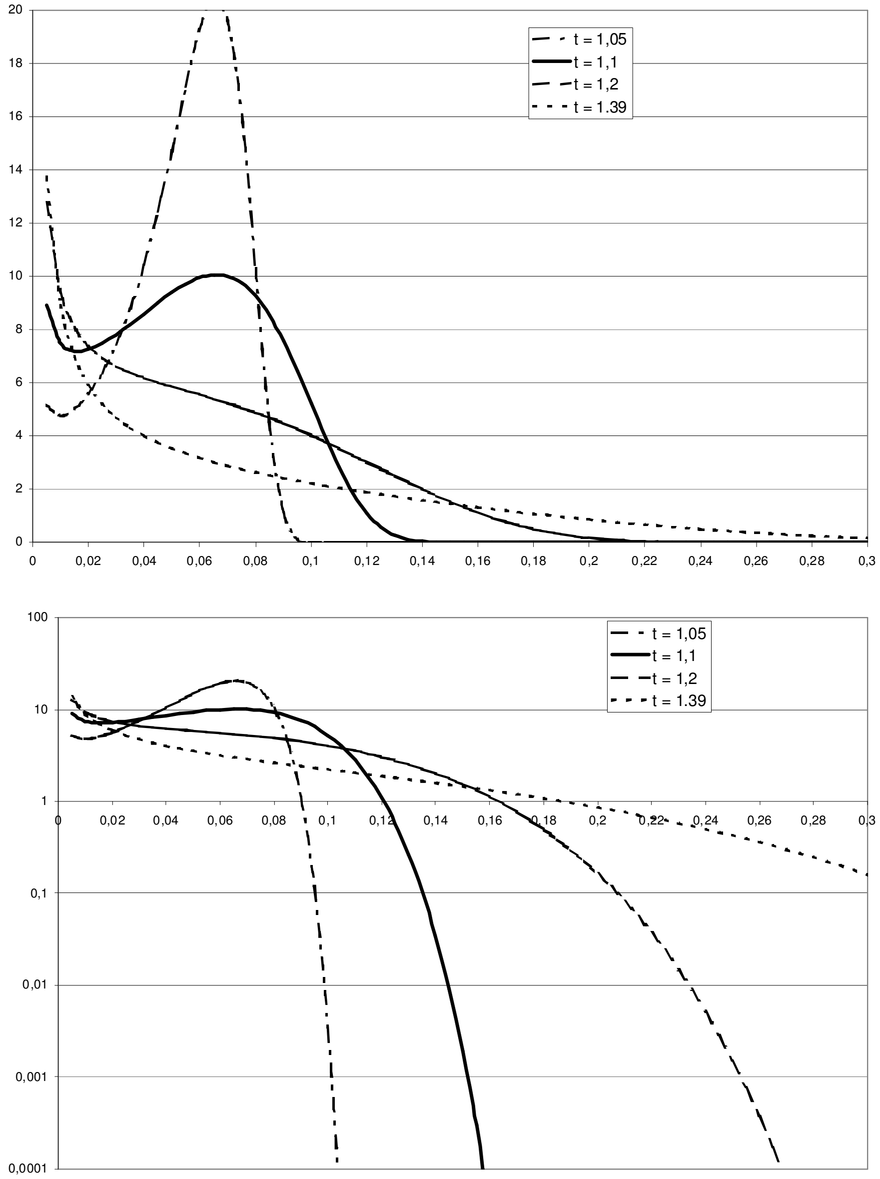

FIGURE 5. Loan loss distributions for the Gumbel model, $p = 5\%$, parameter ($\theta = 1.05, 1.1, 1.2, 1.39$). ($\theta = 1$ corresponds to independence, dependence increases with θ.)

can be seen nicely at the plot for $\theta = 1.1$. As dependency is further increased, the peak around $q = 5\%$ disappears, and in the end the distribution slowly approaches the perfect positive dependence limit with mass only at $q = 0$ and $q = 100\%$.

5.4. **Comparison for Constant Bivariate Default Correlation.** If the default correlation (i.e. the linear correlation coefficient) between two default events

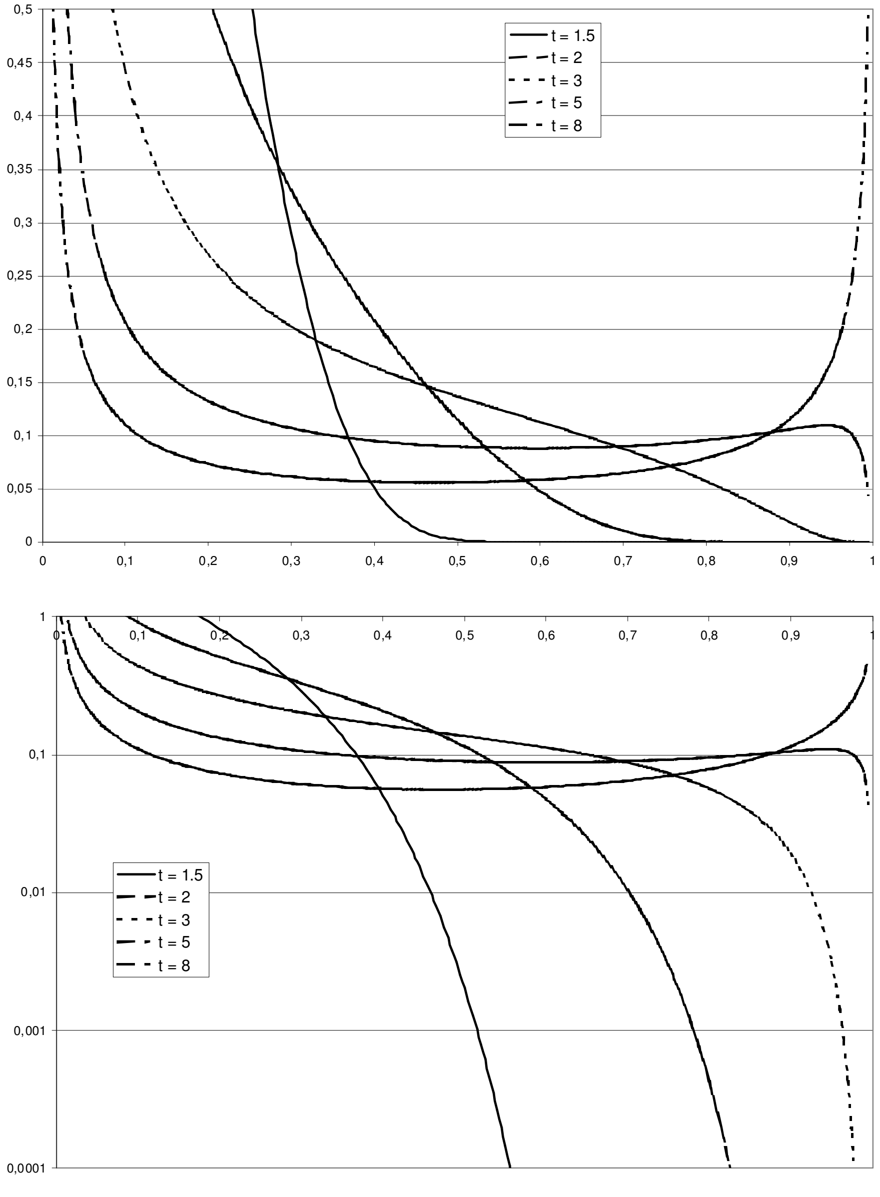

FIGURE 6. Loan loss distributions for the Gumbel model, $p = 5\%$, large dependency parameter ($\theta = 1.5, 2, 3, 5, 8$).

is fixed at $\rho = 10\%$, we have to choose the following model parameters: $\varrho_V = 30.55\%$ in the Vasicek-model, $\theta = 18.12\%$ in the Clayton model, and $\theta = 1.39$ in the Gumbel model. This level of bivariate default correlation corresponds to a high, but not unrealistic level of default correlation in the loan portfolio.

In general, just specifying the default correlation between any *two* obligors does not completely determine the loss distribution of the *whole* portfolio. There are many more defaults involved in the events that we are considering here. Even in a portfolio of just 100 obligors, a 10% loss rate would amount to 10 defaults, and it requires a lot of faith to assume that the joint default probabilities of any two obligors gives us much information on the probability of this event. (It does give some information, though.) For the comparison, we used the default correlation primarily to fix the last remaining parameter in the models, and we are interested how large the differences between the different models may be.

Figure 7 shows the result of the model comparison. There are two surprises: First, the Gaussian (Vasicek) model and the Clayton model imply almost identical loss distributions, and it seems that for these models, the bivariate default correlations do have very similar implications for the loss distribution.

The second surprise is the large deviation of the Gumbel model from the other two. The Gumbel model has significantly more probability mass for losses between 10% and 30% of the portfolio (and for zero losses: the density is infinite there), but then again it has significantly less probability mass for higher default events (losses larger than 30%). This result cannot be driven by any correlation-type measure that only measures the dependency between *two* obligors' defaults, it is driven by higher order moments.

Finally, we should mention that the Frank copula would imply a loss distribution that is yet again fundamentally different. The mixing variable in the Frank copula only takes values on the positive integers. Thus, the large portfolio losses can only take a countable number of values. The loss distribution will be discrete, with discrete steps for different values of Y.

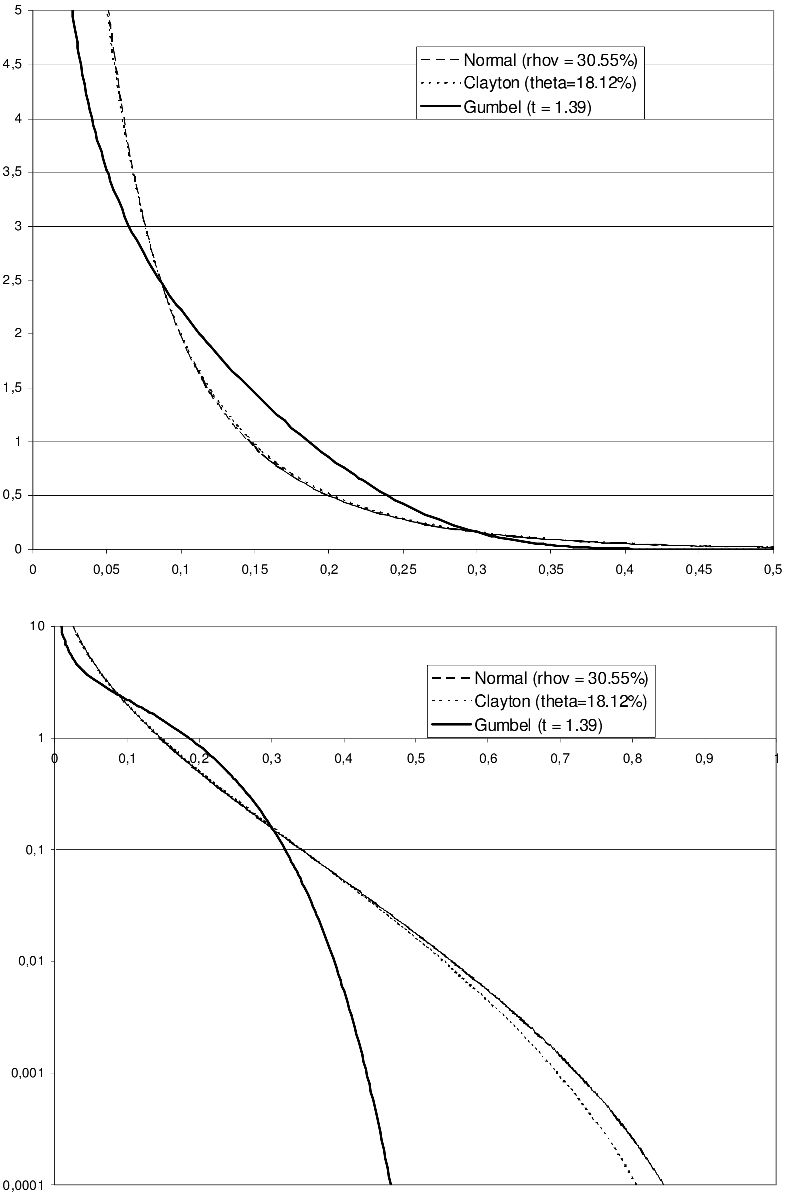

Figure 7. Loan loss distributions for the Gaussian (Vasicek), the Clayton and the Gumbel model. Default correlation between two default events is fixed at $\rho = 10\%$. Individual default probability $p = 5\%$. The parameter values are $\varrho_V = 30.55\%$ (Vasicek); $\theta = 18.12\%$ (Clayton); and $\theta = 1.39$ (Gumbel)

6. Conclusion

This paper has shown three things: First, modelling joint distributions in a different way than just using a variant of the multivariate normal distribution function is feasible. In particular, there are algorithms (such as the one by Marshall and Olkin (1988)) that allow the efficient generation of dependent random numbers in high dimensions. We gave a few examples, but the class of Laplace transforms of positive random variables (and thus of possible dependency structures) that can be generated with the Marshall and Olkin (1988)-algorithm is much larger.

Second, it is worthwhile to investigate the effect of the implicit assumption of a Gaussian dependency structure on the risk measures and the returns distribution of the portfolio. As we have seen in the credit risk case, this effect can be either minor (if one only compares the Vasicek model to the Clayton-dependent model) or significant (if one thinks the Gumbel copula is a realistic alternative).

And finally, we have provided an application of this modelling strategy to the field of credit risk modelling. Credit risk is a particularly interesting application because here the consequences of extreme events are large, and much less data is available than for example for equity returns. Yet, the simple 0-1 structure of default-survival allowed us the derivation of some closed-form solutions for the loss distributions of large portfolio loan losses, and we could compare the implications of these models without having to resort to lengthy simulations.

References

Devroye, Luc (1986). *Non-uniform random variate generation.* Springer, Berlin, Heidelberg, New York.

Frey, Rüdiger and Alexander J. McNeil (2001). “Modelling dependent defaults.” working paper, Department of Mathematics, ETH Zürich.

Gupton, Greg, Christopher Finger, and Bhatia Mike (1997). “Credit metrics - technical document.” Technical document, Risk Metrics Group.

Joe, Harry (1997). *Multivariate Models and Dependence Concepts*, vol. 37 of *Monographs on Statistics and Applied Probability.* Chapman and Hall, London, Weinheim, New York.

Li, David. X. (2000). “On default correlation: A Copula function approach.” working paper 99-07, Risk Metrics Group.

Marshall, Albert W. and Ingram Olkin (1988). “Families of multivariate distributions.” *Journal of the American Statistical Association*, 83, 834–841.

Merton, Robert C. (1974). “On the pricing of corporate debt: The risk structure of interest rates.” *Journal of Finance*, 29, 449–470.

Nelsen, Roger B. (1999). *An introduction to copulas*, vol. 139 of *Lecture Notes in Statistics*. Springer, Berlin, Heidelberg, New York.

Schönbucher, Philipp J. and Dirk Schubert (2001). "Copula-dependent default risk in intensity models." Working paper, Department of Statistics, Bonn University.

Vasicek, Oldrich (1987). "Probability of loss on loan portfolio." Working paper, KMV Corporation.

Vasicek, Oldrich (1997). "The loan loss distribution." Working paper, KMV Corporation.

Author's address:
Philipp J. Schönbucher
Department of Statistics, Faculty of Economics, Bonn University
Adenauerallee 24–42, 53113 Bonn, Germany,
Tel: +49 - 228 - 73 92 64, Fax: +49 - 228 - 73 50 50

E-mail address: `P.Schonbucher@finasto.uni-bonn.de`

URL: `http://www.finasto.uni-bonn.de/~schonbuc/`