

Optimal Credit Limit Management Under Different Information Regimes*

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Abstract

Credit limit management is of paramount importance for successful short-term credit-risk management, even more so when the situation in credit and financial markets is tense. We consider a continuous-time model where the credit provider and the credit taker interact within a game-theoretic framework under different information structures. The model with complete information provides decision-theoretic insights into the problem of optimal limit policies and motivates more complicated information structures. Moving to a partial information setup, incentive distortions emerge that are not in the bank's interest. We discuss how these distortions can effectively be reduced by an incentive-compatible contract. Finally, we provide some practical implications of our theoretical results.

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In this paper we are concerned with the optimal management of credit limits. For a bank, it is indispensable to have implemented a sound risk management concept, which helps to steer the bank through tense market situations. The drastic stock market downturns in the recent years aggravated the deterioration of the credit markets. As a consequence, the managing of credit risk shifted its focus from a long or mid-term horizon to a very short-term horizon. For credit risk management, short-term policies are mainly concerned with managing credit limits. The credit limit is an approved level of credit allowable that ideally should be compatible with the financial status of the bank's debtor. One of the functions of the bank's credit department is to ensure the proper control of a debtor's account. The credit limit provides a means whereby one aspect of control may be achieved by imposing an upper bound on the potential credit exposure. In the short-term, the management of credit limits is largely the only instrument to control and bound credit losses.

Examples highlighting the importance of proper limit management are numerous. In the aftermath of the LTCM crisis, the United States General Accounting Office noted in their Report to Congressional Requesters of October 1999 that some of LTCM's creditors and counterparties failed to apply appropriate prudential standards. According to the President's Working Group report, such standards include also the management of credit limits on counterparty exposures.

In late 2002, ANZ identified its structured finance division as the major source of the bank's bad debts, as large corporates like Enron collapsed the year before. Most of the losses were related to associated lending rather than the structured finance projects themselves. Consequently, ANZ was "de-risking" their corporate portfolio mostly by lowering credit limits on large single customer exposures. In December 2001, UBS and Credit Suisse offered SWISS, one of many troubled airline companies, a credit limit of CHF500m for restructuring their business. Against the backdrop of the war in Iraq and the spreading of SARS in Asia, outlook for economic growth darkened, in particular for the airline industry. As a result, in April 2003 UBS and Credit Suisse cut their credit limit down to SWISS's actual credit usage, which was by then at CHF100m.

The goal of this paper is to analyze optimal limit management when *a)* the bank has complete information about the company's surplus, *b)* the bank has only partial information about the company's surplus, and *c)* the bank uses an incentive-compatible contract that induces the company to put some effort into revealing information about its true surplus.

The paper is organized as follows. The next section cites the studies that provide background information on credit risk modeling, and defines the position of our work in the literature. Section 2 introduces the notation and presents the model with complete information on the debtor's surplus. In Sec-

tion 3, we modify the complete information model by assuming only partial information on the debtor's state variable. Since partial information induces some undesirable effects, we introduce in Section 4 an incentive-optimal contract to reduce these effects. The practical implications of the theoretical results are discussed in Section 5. Section 6 concludes.

1 Background

In recent years, credit risk management has attracted a lot of attention from the academic community. Three main directions of research in quantitative credit risk management emerged. The first stream analyzes credit portfolios from a diversification point of view. See, e.g., Lucas (1995), Li (2000), Giesecke and Weber (2002), Egloff, Leippold and Vanini (2003), for a recent account. A principal goal is to define reasonable risk and diversification measures and, finally, to determine optimal allocations. A second branch studies the risk transfer of credit positions by either financial or actuarial contracts. Of major concern is the design and valuation of such credit risk contracts. Francis, Frost and Whittaker (1999) and Nelken (1999) provide an exhaustive description of the credit derivative industry. The third stream of literature is focusing on the securitization of credits and loans. See, e.g., Das (2000) among others.

All the above approaches to managing credit risk share a common feature: They can be used to optimize a credit exposure on a mid-term or a long-term basis. Therefore, for all institutions which did not foresee the recent turbulent times in the credit markets, the above instruments were not of much use. E.g., restructuring the balance sheet by transferring credit risk through credit derivatives became almost impossible or, at least, very expensive. When risk cannot be transferred, the focus naturally changes to controlling the current risk exposure on a short-term basis by managing credit limits. During a stock market downturn, the relevance of such a strategy becomes even more accentuated, since heavy losses in stock market values usually lead to pronounced liquidity problems for the bank's credit clients. In many cases, a decrease in the debtor's equity triggers an increase in the demand for credit and loans. Therefore, if banks are not attentive to their credit exposure, it is likely to grow rapidly. By focusing on limit management, we illuminate an aspect of credit risk modelling different from the traditional approaches initiated by Merton (1974) and Black and Cox (1976). In particular, we isolate our analysis from the possibility of default. For the main purpose of this paper, the study of the optimal limit policy under different information regimes, neglecting default is not as severe as it might seem. In practice, the cost of default is often subsumed in the price per unit of credit supplied to the company. In particular, this price is determined by an appropriate margin and several cost components. Taking the perspective of the bank's credit department, these cost components are

comprised by *a*) the internal interest rate owed to the treasury department, *b*) the internal production costs, *c*) the regulatory costs, *d*) the costs determined by the internal credit rating. Thus, the company’s default probability enters the price per unit of credit supply through the internal credit rating. For a debtor with high default probability, the bank will adjust its price for providing credit accordingly. In our model setup, this price enters the optimization problem of the bank. Therefore, to understand the effect of default, at least in part, we can trace out the corresponding comparative static.

From a modelling point of view, we assume that the risk and return characteristics of the debtor’s investment process affect the bank’s limit assessment decision at any given time. The demand for credit following from the optimality of the debtor’s investment decision defines an earning component in the bank’s value function. In turn, the bank’s limit assessment affects the optimization problem of the firm by bounding the possible credit exposure. Therefore, the analysis of credit limit management is defined as the solution of a dynamic non-cooperative game. This setup relates our model to the theory of differential games, first introduced in Isaacs (1954). In particular, our model comes close to the continuous-time model in Holmström and Milgrom (1987), where one agent controls the drift rate vector of a multi-dimensional Brownian motion. However, the model that we present differs in the underlying information structure. In addition to incomplete information on the firm’s actions, our model features partial information on the state variable. Furthermore, the debtor’s credit decision influences both drift and variance of the surplus.

2 Limit Policy with Complete Information

In this section, we consider a model with complete information.¹ We define a financial market with a terminal time T on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Uncertainty is modelled by W_t , a one dimensional Brownian motion. Our economy is populated by a company C representing the credit demand side. The credit supply is provided by a bank B . We assume that both players B and C are rational and maximize expected utility over a period $[t, T]$, $0 \leq t < T$, with T possibly equal to infinity.

The company C maximizes expected surplus given an upper limit for the surplus’ variance. The surplus, denoted by S_t , is defined as the difference between assets and liabilities and serves as the state variable in our model. The company (debtor) chooses the optimal credit amount to maximize the expected surplus. This credit demand is expressed in terms of a fraction c_t of the current surplus S_t , i.e., the credit demand in absolute terms equals $c_t S_t$.

¹In practice, lombard loan relationships can be casted into such a model setup.

The bank, in turn, chooses the optimal limit policy ℓ_t to maximize earnings minus costs from credit lending. Hence, in our model, the choice variables are *a)* the credit demand c_t as a fraction of the surplus S_t , and *b)* the credit limit ℓ_t provided by the bank.

If the company does not demand any credit, the surplus is assumed to evolve as

$$dS_t = \mu_0 dt + \sigma_0 dW_t.$$

As soon as the company demands credit, we assume that these additional resources are invested in a project with different return dynamics, such that the stochastic differential equation (SDE) for the surplus changes to

$$dS_t = (\mu_0 + (\mu_1 - p)c_t) dt + (\sigma_0 + \sigma_1 c_t) dW_t.$$

The fraction of surplus c_t affects the future surplus two-fold. First, it changes the drift of the surplus. The money borrowed by the firm is invested to obtain a return on the new investment, μ_1 , that may differ from the actual one, μ_0 . From the new return on investment we subtract the price $p > 0$ paid for one unit of the loan. Second, investing in new projects also alters the risk of the present business, which is reflected by the additional volatility parameter σ_1 . Therefore, by lending money, the company can increase the mean of the surplus, but at the same time the company increases its surplus volatility.

We denote by $J^C(S, t)$ the value function of company C and assume that C solves the following optimization:

$$(C') \quad \left\{ \begin{array}{l} J^C(S, t) = \max_{c_t} \mathbb{E} \left[\int_t^T e^{-\delta(s-t)} S_s ds \mid \mathcal{F}_t \right], \\ \text{s.t.} \quad \int_t^T \text{Var} \left[e^{-\delta(s-t)} S_s \mid \mathcal{F}_t \right] ds \leq \zeta^2, \\ dS_t = (\mu_0 + (\mu_1 - p)c_t) dt + (\sigma_0 + \sigma_1 c_t) dW_t, \\ c_t S_t \leq \ell_t, \quad 0 \leq c_t, \quad \forall t \in [0, T]. \end{array} \right.$$

We henceforth abbreviate the company's set of constraints by \mathcal{C} . The optimization problem (C') mimics the single-period mean-variance approach of Markowitz (1952). However, contrary to Markowitz's static model, we formulate the optimization problem in continuous time. Furthermore, instead of maximizing the end-of-period surplus subject to an end-of-period variance bound, we assume that the company C is concerned with the regularity and smoothness of surplus evolution. Indeed, most firms have an incentive to smooth the variation of the surplus. If surplus is too erratic over time, firms face difficulties in explaining their course of business to investors, the public, and other stakeholders, and finally to the banks. Furthermore, smoothing may result from a tax minimizing strategy as originally documented in Lintner (1956). Therefore, the firm maximizes the expected surplus rate over the time horizon $T - t$ given a variance bound. Thus, the risk has to be smaller

than a given acceptance level along the whole optimal investment path and not only at the investment horizon. In addition to the variance constraint, the inequality constraint $c_t S_t \leq \ell_t$ puts an upper bound on the credit demand c_t expressed as a fraction of the current surplus. This amount cannot be larger than the amount of the credit limit ℓ_t . Finally, we assume that $c_t \geq 0$.

Independent of the bank's decision problem, the optimization problem of company C is difficult to solve in the form (\mathbf{C}') , since it is not separable in the dynamic programming sense. The non-separability can be circumvented as follows: Independent of the dynamics of the surplus and other linear constraints, the company's preferences in (\mathbf{C}') are equivalent to

$$J^C(S, t, \omega) = \max_{c_t \in \mathcal{C}} \left(\int_t^T \mathbb{E} [e^{-\delta(s-t)} S_s | \mathcal{F}_t] - \omega \text{Var} [e^{-\delta(s-t)} S_s | \mathcal{F}_t] ds \right) \quad (1)$$

with \mathcal{C} the same feasible set as in (\mathbf{C}') . The parameter ω reflects the trade-off between return and risk. Now, consider the separable problem $(\mathbf{C1})$, which is a two-parameter model ω, λ :

$$(\mathbf{C1}): \quad J^C(S, t, \omega, \lambda) = \max_{c_t \in \mathcal{C}} \mathbb{E} \left[\int_t^T e^{-\delta(s-t)} (\lambda_s S_s - \omega S_s^2) ds | \mathcal{F}_t \right].$$

Then, the following result holds:²

Proposition 1. *If c^* is a solution to problem (\mathbf{C}') , and if the surplus follows a linear SDE of the form,*

$$dS_t = (a_t + b_t S_t) dt + (\alpha_t + \beta_t S_t) dW_t, \quad a_t + \alpha_t \beta_t \geq 0,$$

then it is also a solution to problem $(\mathbf{C1})$ with $\lambda_t^ = 1 + 2\omega c_t^*$, where S^* is the wealth trajectory corresponding to c^* .*

We first remark that the inequality part of the sufficient condition in Proposition 1 is not as restrictive as it might seem. In our present model, it is reasonable to assume positive values for the parameters a_t and α_t . Then, if β_t obtains a negative value, this would mean that a higher surplus and an increase in credit usage would reduce the instantaneous surplus variance. Since such a setup is counterintuitive, we can safely assume that $a_t + \alpha_t \beta_t \geq 0$ holds.

Having presented the company's C setup, we next consider the banks de-

²For a proof, we refer to Zhou and Li (2000) and Leippold, Trojani and Vanini (2004) in a similar setting.

cision problem **(B)**. It reads

$$\text{(B1)} \quad : \quad \begin{cases} J^B(S, t) = \max_{\ell_t} \mathbb{E} \left[\int_t^T e^{-\delta(T-s)} (pc_s S_s - \kappa \ell_s) ds \mid \mathcal{F}_t \right], \\ s.t. \quad \ell_t \geq 0, \\ dS_t = (\mu_0 + (\mu_1 - p)c_t) dt + (\sigma_0 + \sigma_1 c_t) dW_t. \end{cases}$$

Following **(B1)**, the bank chooses the limit amount ℓ_t over the duration $T - t$ such that expected earnings from lending money to company C minus the capital costs κ of providing limit ℓ_t are maximized. The costs of economic capital are calculated on the limit amount set off for the client and not on the actual credit exposure. If payment were on the credit exposure, the decision problem would become trivial: The bank would fulfill any credit request of the company C as long as $p > \kappa$. The assumption of a constant κ reflects the situation of a bank applying the Basel II's standard approach. If the bank would use an internal model approach, κ would not be constant over different limit amounts. However, for our analysis we keep κ constant.

Since the decision variables of each player affects the value function of the other one and the variables are functions of time, the problems **(C1)** and **(B1)** define a non-cooperative game. To obtain a solution we apply the concept of a subgame-perfect Nash equilibrium: A pair (c^*, ℓ^*) is a Nash equilibrium, if

$$J^B(S, t, c^*, \ell^*) \geq J^B(S, t, c^*, \ell) \text{ , } J^C(S, t, c^*, \ell^*) \geq J^C(S, t, c, \ell^*)$$

for any feasible policies c and ℓ and where c^* (ℓ^*) is the optimal strategy of the company (bank).

Before trying to solve the game defined above, the following argument simplifies the analytics essentially. We claim that for the equilibrium strategy $c_t^* = \frac{\ell_t^*}{S_t}$ holds. To see this, we denote by v_t the bank's instantaneous utility at time t . Suppose that for $\epsilon > 0$, $c_t^* S_t^* + \epsilon = \ell_t^*$ is a Nash equilibrium. Then,

$$v_t^\epsilon = pc_t S_t - \kappa \ell_t = p(\ell_t^* - \epsilon) - \kappa \ell_t = (p - \kappa)\ell_t - p\epsilon = v_t - p\epsilon \text{ .} \quad (2)$$

Therefore, the bank is always better off choosing a limit policy with $\epsilon = 0$ instead of $\epsilon > 0$. In other words, we have only to solve one optimization problem. The optimality for the other player then follows at once.

Proposition 2. *Consider an economy with two players solving **(C1)** and **(B1)**, respectively. The strategies*

$$c_t^* = \frac{\ell_t^*}{S_t} \text{ , } \ell_t^* = \gamma_1 S_t + \gamma_2 S_t^2,$$

are a subgame perfect Nash equilibrium, where

$$\gamma_1 = \frac{\sqrt{\delta}}{\sigma_1}, \quad \gamma_2 = \frac{\mu_0(\mu_1 - p) - \delta\sigma_0\sigma_1}{\sqrt{\delta}\sigma_1(\mu_1 - p + \sqrt{\delta}\sigma_1)}.$$

From Proposition 2 it follows that the value functions of the bank and the company in equilibrium are quadratic functions in S_t .

Proposition 3. *Consider an economy with two players solving (C1) and (B1), respectively. The value function of the bank reads*

$$J^B(S, t) = e^{-\delta(T-t)}(p - \kappa) \left(b_0 + b_1 S_t + \frac{1}{2} b_2 S_t^2 \right), \quad (3)$$

and the value function of the company is given by

$$J^C(S, t) = e^{-\delta(T-t)} \left(k_0 + k_1 S_t + \frac{1}{2} k_2 S_t^2 \right), \quad (4)$$

where the constants $b_i, k_i, i = 1, 2, 3$, are given in the Appendix, equations (A.1) to (A.3), and (A.4) to (A.6), respectively.

From Proposition 2 we further note that the optimal credit demand c_t^* is an affine function of the surplus S_t with a marginal derivative of γ_2 . It turns out that the sign of this derivative characterizes the equilibrium solution of the two player economy. The proposition also shows that it may be optimal for both the bank and the company to terminate their credit relation even if the surplus is positive. This is the case when the value of S equals $S = -\gamma_1/\gamma_2$. This ratio can only become positive if $\gamma_2 < 0$. With $\gamma_2 < 0$, the optimal limit policy ℓ^* is concave in S . The sign of γ_2 not only determines whether the optimal limit policy is either convex or concave in S , but also determines the statistical properties of the surplus dynamics. More precisely, it determines whether S follows a stationary or a non-stationary process. Using the optimal policies in the surplus dynamics, the surplus is stationary if and only if

$$(\mu_1 - p)\gamma_2 > 0.$$

Figure 1 plots the optimal limit ℓ^* as a function of the company's surplus S . When the riskiness of the company's investments is relatively small compared to the corresponding returns, we get $\gamma_2 > 0$ and the optimal limit is a convex function of the surplus. For $\gamma_2 = 0$, the optimal limit is just a straight line with slope γ_1 . For $\gamma_2 < 0$, the optimal limit as a function of surplus becomes concave. In Panel (A) of Figure 2, we plot a possible trajectory of the surplus. We consider the cases $\gamma_2 > 0$ (dotted line, company 1) and $\gamma_2 < 0$ (solid line).

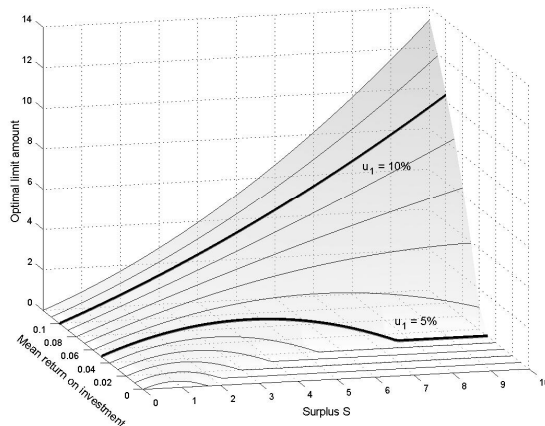


Figure 1: Optimal limit policy. We make the following assumptions: $\mu_0 = 2\%$, $\sigma_0 = 10\%$, $\sigma_1 = 30\%$, $\delta = 5\%$, $p = 1\%$. In order to generate optimal limit policies where γ_2 switches sign, we set $0 < \mu_1 < 12\%$. As examples, we highlight the optimal policy for $\gamma_2 > 0$ by assuming $\mu_1 = 10\%$, and the optimal policy for $\gamma_2 < 0$ by assuming $\mu_1 = 5\%$.

As we see from Panel (A), the trajectories for S_t when $\gamma_2 > 0$ and $\gamma_2 < 0$ are almost indistinguishable, at least for the first five years. However, the stationarity property has a strong effect on the optimal limit policy of the bank. In Panel (A), we add the relative credit demands for the two cases $\gamma_2 > 0$ and $\gamma_2 < 0$. Comparing the growth of the credit demand and the surplus that may serve as an indicator of the company's health, we see that the company demands credit in a procyclical manner if $\gamma_2 > 0$ and anticyclical if $\gamma_2 < 0$. Therefore, when the surplus is decreasing, a company that has not the opportunity to invest in a project with high expected return, i.e., μ_1 is small, reduces its relative credit demand. Furthermore, since the bank decreases the credit limits, the company is locked out of new investment. On the other side, a company that can invest in highly profitable projects, i.e., μ_1 is large, tends to invest more in relative terms when the economy goes down.

Panel (A) of Figure 2 plots the paths of the optimal limit policies and their corresponding surplus when $\gamma_2 > 0$ (dotted line) and $\gamma_2 < 0$ (solid line). When the surplus is a non-stationary process, the optimal limit process lies considerably above the optimal limit when the surplus process is stationary. In Panel (B), we also plot the bank's profit from providing credit limits. Since with $\gamma_2 > 0$ the company has an increased demand for credit limits, the bank makes a larger profit than if we had $\gamma_2 < 0$. From an econometric viewpoint, it is often hard to give a conclusive statement about the stationarity property of a process. This difficulty leads us directly to the next question: How do the

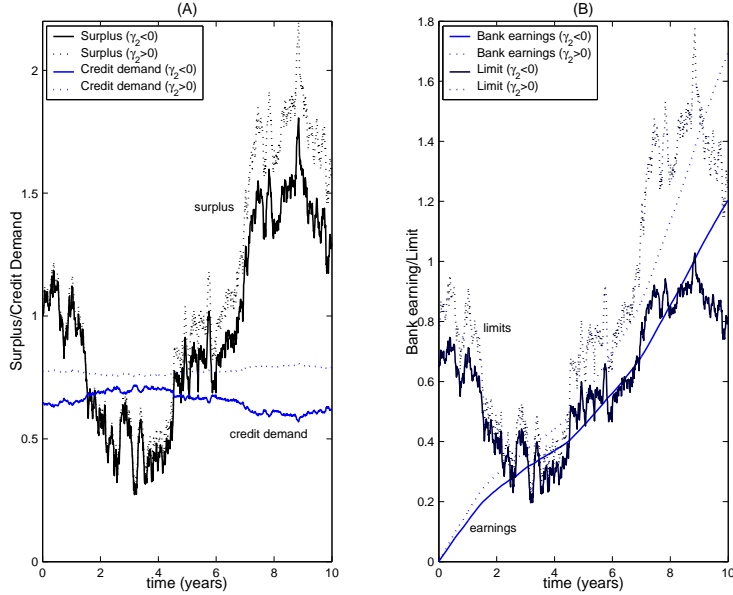


Figure 2: Optimal limit policy. We make the following assumptions: $\mu_0 = 2\%$, $\sigma_0 = 10\%$, $\sigma_1 = 30\%$, $\delta = 5\%$, $p = 1\%$, $\kappa = 0.8\%$. In order to generate $\gamma_2 < 0$, we set $\mu_1 = 5\%$. For $\gamma_2 > 0$, we simply set $\mu_1 = 10\%$. Panel (A) simulates one trajectory of the surplus given $\gamma_2 > 0$ (dotted line) and $\gamma_2 < 0$ (solid line) and its corresponding relative credit demand c_t . In Panel (B), we plot the absolute optimal limit policy corresponding to the surplus dynamics in Panel (A) for $\gamma_2 > 0$ (dotted line) and $\gamma_2 < 0$ (solid line), respectively. In Panel (B), we also plot the bank’s (cumulative) profit resulting from the company’s credit demand.

optimal policies change, when the bank has only partial information about the surplus and its dynamics?

3 Limit Policy with Partial Information

In practice, bank B often has only partial information about the current surplus of company C .³ Due to inaccuracies in the bank’s measurement of the true surplus, the bank cannot measure S_t itself, but a disturbed version of it:

$$\zeta_t = S_t + \text{“noise”}.$$

The bank has to make a credit limit assessment given a best estimate \hat{S} of the company’s surplus based on the observed signal ζ_t . Such a best estimate is obtained by using a Kalman-Bucy filter (see Kalman (1960) and Kalman

³Such a model setup is relevant, e.g., if we interpret $c_t S_t$ as a deposit credit amount.

and Bucy (1961) for the original contributions). Intuitively, the bank filters away the noise from the system in an optimal way. To formalize the bank's behavior, we assume that the dynamics of the signal as observed by the bank follows the stochastic differential equation

$$\text{(O)} \quad : \quad d\zeta_t = (A_0 + A_1 S_t) dt + B dZ_t.$$

The signal ζ_t can comprise such things as analysts' reports, share prices, press releases and so on. The state variable, i.e., the true surplus S_t of company C , evolves according to

$$\text{(U)} \quad : \quad dS_t = (\mu_0 + (\mu_1 - p)c_t^*) dt + (\sigma_0 + \sigma_1 c_t^*) dW_t,$$

with c_t^* the company's optimal credit decision taking the bank's decision ℓ^* into account. We note that ℓ^* is now a function not only of S_t , but also of ζ_t , the signal received by the bank. Then, company C solves

$$\text{(C2)} : \quad J^C(S, t, \omega, \lambda) = \max_{c_t \in \hat{\mathcal{C}}} \mathbb{E} \left[\int_t^T e^{-\delta(s-t)} (\lambda S_s - \omega S_s^2) ds \mid \mathcal{F}_t \right]. \quad (5)$$

The above optimization problem is the same as the one in **(C1)**, with a subtle difference in the constraint set $\hat{\mathcal{C}}$: The upper bound for the current usage $c_t \leq \ell_t/S_t$ depends now also on the signal ζ_t through ℓ_t . Using the same arguments as in Section 2, the optimal policy of company C given the limit ℓ_t is

$$c_t^* = \ell_t/S_t. \quad (6)$$

To derive the bank's optimal limit policy, we first have to elaborate on the signal process and the best estimate for company's C true surplus. To simplify our exposition, the two Brownian motions W_t and Z_t are assumed to be independent. We note that \hat{S}_t is \mathcal{G}_t -measurable, where \mathcal{G}_t is the σ -algebra generated by the Brownian motion Z_t . By saying that \hat{S}_t is the "best guess" of the surplus we mean that

$$\int_{\Omega} |S_t - \hat{S}_t|^2 d\mathbb{P} = \mathbb{E} \left[|S_t - \hat{S}_t|^2 \right] = \inf_Y \left\{ \mathbb{E} \left[|\hat{S}_t - Y|^2 \right] \mid Y \in L^2(\mathcal{G}, \mathbb{P}) \right\}. \quad (7)$$

The equation for the unobservable surplus **(U)** and the equation for the observable signal **(O)** define a nonlinear optimal filtering problem. Deriving closed-form solutions for such filters is generally not possible. However, we can make use of the fact that the conditional distribution $F_{\zeta_t} = \mathbb{P}(S_t \leq k \mid \mathcal{G}_t)$ is \mathbb{P} -Gaussian.

Proposition 4. *Given equations (U) and (O), the process \hat{S}_t satisfying (7) is*

$$d\hat{S}_t = \left(\mu_0 + (\mu_1 - p)\hat{c}_t + \rho_t \frac{A_1^2}{B^2}(S_t - \hat{S}_t) \right) dt + \rho_t \frac{A_1}{B} dZ_t, \quad \hat{S}_0 = \mathbb{E}[S_0], \quad (8)$$

with ρ_t the solution to the Riccati equation

$$\frac{d\rho_t}{dt} = (\sigma_0 + \sigma_1 \hat{c}_t)^2 - \rho_t^2 \frac{A_1^2}{B^2}, \quad \rho_0 = \mathbb{E}[(\hat{S}_0 - S_0)^2].$$

For a proof of the above proposition, we refer to Theorem 12.1 of Liptser and Shiryaev (2001). Equation (8) deserves some explanation. Recall that in Section 2 we define c_t as a fraction of current surplus S_t . The product $c_t S_t$ makes up the amount of credit demand in absolute terms. Here, we claim that S_t cannot be observed by the bank. However, what the bank observes is the actual credit demand $c_t S_t$. If the bank would already know the true value of c_t , then the bank would know the true value of S_t . To avoid such a trivial setting, we have to assume that not only S_t is unobservable, but also c_t . Therefore, when the bank is faced with the credit demand $c_t S_t$, the bank first forms a best estimate \hat{S}_t about the true surplus. From this estimate, the bank then infers the estimated value \hat{c}_t , since we require $c_t S_t = \hat{c}_t \hat{S}_t$. (See Figure 3.) It follows that in equation (8) the value of \hat{c}_t enters the drift of $d\hat{S}_t$ and not the value of c_t . Furthermore, the amount $\hat{c}_t \hat{S}_t$ enters into the optimization problem (B2) below. Therefore, at first sight, it might look as if the bank is not concerned about the true value of \hat{c}_t and \hat{S}_t . However, the dynamics of \hat{S}_t enter the constraint set $\hat{\mathcal{B}}$. Hence, the optimization problem in (B2) is indeed different from the optimization problem (B1):

$$(B2): \quad J^B(\hat{S}, t) = \max_{\ell_t \in \hat{\mathcal{B}}} \mathbb{E} \left[\int_t^T e^{-\delta(T-s)} (p \hat{c}_s \hat{S}_s - \kappa \ell_s) ds \mid \mathcal{G}_t \right], \quad (9)$$

where $\hat{\mathcal{B}}$ is same constraint set as for problem (B1), but with the surplus S_t replaced by its best estimate \hat{S} as given in equation (8).

The Hamilton-Jacobi-Bellmann (HJB) equation for (B2) is difficult to solve and possibly no closed-form solution can be obtained. One route to take would be to use numerical methods to obtain the optimal limit policy. However, at this stage we are not interested in quantitatively exact results, but want to learn more about qualitative features of the model. To this end, we use perturbation theory and expand around a point that has a concrete economic interpretation: Assume that bank B is competent and the estimates of the surplus do not deviate drastically from the true surplus, at least in relative terms. Then $\left| \frac{S_t - \hat{S}_t}{S_t} \right|$ is small and close to 0. Moreover, as ρ_t is highly non-

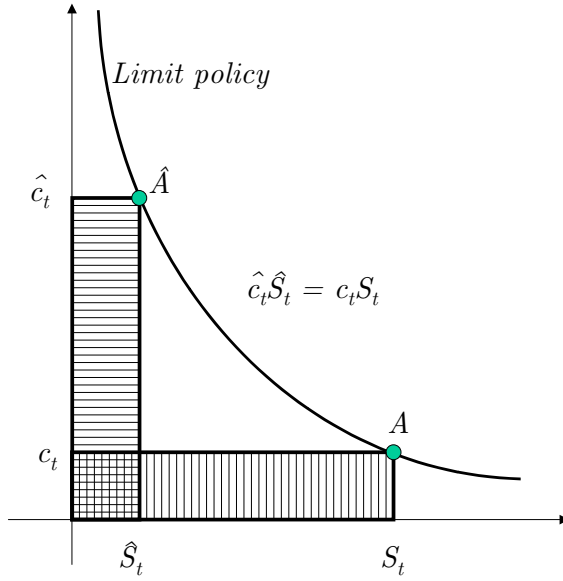


Figure 3: Credit demand: The bank observes the absolute value of the credit demand. This amount equals $c_t S_t = \hat{c}_t \hat{S}_t$. However, neither the true value of the surplus, S_t , nor the fraction c_t are observable.

linear in c_t , we work with a first-order approximation in $c_t \sigma_1$. The results for the approximative strategies $c_t := c_t^{(1)*} + O\left(\left|\frac{S_t - \hat{S}_t}{S_t}\right|, (c_t \sigma_1)^2\right)$ and $\ell_t := \ell_t^{(1)*} + O\left(\left|\frac{S_t - \hat{S}_t}{S_t}\right|, (c_t \sigma_1)^2\right)$ are given in the next proposition.

Proposition 5. *Consider an economy with two players solving (B2) and (C2), where player B has only partial information on C's surplus. With B using ζ_t as a signal for the company's current surplus S_t , the strategies*

$$c_t^{(1)*} = \frac{\ell_t^{(1)*}}{S_t}, \quad \ell_t^{(1)*} = \hat{\gamma}_1(t) \hat{S}_t + \hat{\gamma}_2(t) \hat{S}_t^2,$$

are an asymptotic subgame perfect Nash equilibrium, where

$$\hat{\gamma}_1(t) = \frac{B\sqrt{\delta}}{r_1(t)A_1}, \quad \hat{\gamma}_2(t) = \frac{\mu_0(\mu_1 - p) - r_0(t)r_1(t)\frac{A_1^2}{B^2}\delta}{\sqrt{\delta}r_1(t)\frac{A_1}{B}\left(\mu_1 - p + \delta r_1(t)\frac{A_1^2}{B}\right)},$$

and $r_0(t)$ and $r_1(t)$ are given in the Appendix, equation (A.7).

From Proposition 5, we see that the differences between γ_i and $\hat{\gamma}_i$ are given

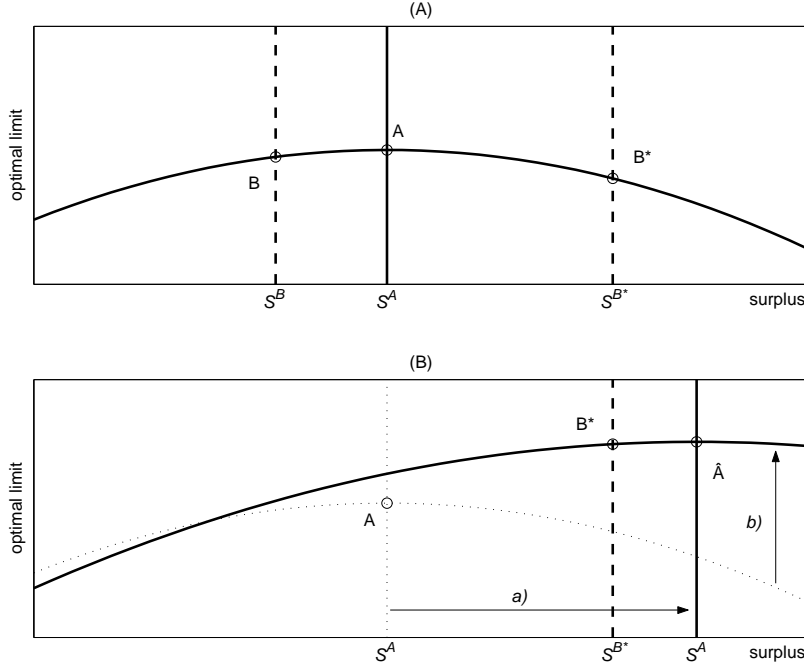


Figure 4: The influence of partial information. We assume: $\mu_0 = 2\%$, $\mu_1 = 5\%$, $\sigma_0 = 10\%$, $\sigma_1 = 30\%$, $\delta = 5\%$, $p = 1\%$. Panel (A) plots the complete information case. The concave curve describes the optimal limit as a function of the surplus. The straight line through point A represents the bank's satiation. The dotted lines represent the company's satiation, i.e., the line through point B when $\omega = 1.2$ and through point B^* when $\omega = 1.125$. Panel (B) plots the case with partial information and for $\omega = 1.125$. For the signal process ζ_t we assume $A_1/B = 10$. The satiation for the bank is moved upwards to \hat{A} . In addition, the optimal limit policy is pushed upwards.

by changes in the volatility terms. More precisely, by substituting

$$\sigma_0 \rightarrow r_0(t) \frac{A_1}{B}, \quad \sigma_1 \rightarrow r_1(t) \frac{A_1}{B}, \quad (10)$$

in the expressions for γ_i we obtain $\hat{\gamma}_i$. The value function of J^B and J^C are again quadratic, but in \hat{S}_t and S_t respectively.

Figure 4 clarifies the influence of first-order partial information in our model. In Figure 4 we plot the optimal limit policy as a function of the surplus S when the mean μ_1 is low such that the optimal limit policy is concave in S .

Panel (A) of Figure 4 plots the case when there is complete information. In addition to the optimal policy (the bold curve), we also plot the different satiation levels at which the company's and the bank's value functions are at

a maximum. Whenever the optimal limit policy crosses a satiation level from the left, the company has no incentives to further increase her surplus, or, on the other hand, the bank has no incentives to further supply additional credit limits. Given the numerical parameter values, the satiation level for the bank in Panel (A) is the bold line that crosses the optimal limit policy in point A. For the company, we plot two different satiation levels. The dashed line that crosses the optimal policy at point B assumes $\omega = 1.2$, whereas the dashed line through point B* assumes $\omega = 1.125$, i.e., in the latter case, the company is slightly less risk-averse. We consider first the case with $\omega = 1.2$. The bank's satiation level is to the right of the company's satiation level. Therefore, the surplus realization will be point S^B , which corresponds to the company's satiation level. The bank will not be able to attain the surplus level that maximizes her value function. However, if the company has a risk aversion coefficient of $\omega = 1.125$, the bank reaches point A. The surplus will be at S^A and, hence, maximizes the bank's value function. The company's satiation level, S^{B^*} , will not be reached. Panel (B) of Figure 4 plots the optimal policy and the satiation levels in case of partial information and for $\omega = 1.125$. For comparison, we also plot the optimal policy and the bank's satiation level given complete information (dotted lines). When we introduce partial information, the volatilities σ_0 and σ_1 change according to equation (10). These changes induce two effects, indicated by the arrows in Panel (B) and labelled accordingly:

- a) A right-shift of the bank's satiation level.
- b) A shift of the optimal limit policy through a decrease in the curve's concavity.

Therefore, the change in the variances σ_0 and σ_1 moves the bank's satiation level from point A to \hat{A} . However, point \hat{A} is to the right of the company's satiation level (point B), that remains unaffected by the introduction of partial information. The resulting surplus, S^{B^*} satisfies now the company's but not the bank's satiation level. Therefore, the company is now better off than in the case with complete information. The bank offers more limit to the company and increases its potential exposure. Furthermore, the bank's regulatory costs (through κ) increase compared to the complete information regime. As a result, the company has a strong incentive to manipulate the signal dynamics ζ . Indeed, by decreasing the ratio A_1/B , the company moves the bank's satiation level to the right. A decrease in A_1/B occurs when the company increases the signal variance B or decreases the A_1 . The parameter A_1 determines the drift component in ζ_t that is proportional to the company's surplus. Therefore, a small A_1 makes the signal ζ_t less liable for predicting the true surplus S_t . We recall that the above results hold to up to first-order in $\left| \frac{S_t - \hat{S}_t}{S_t} \right|$, i.e., the true

surplus is already near the bank’s best guess. Therefore, considering higher-order terms, the distortion effects due to partial information would be even more accentuated.

4 Contracting under Partial and Incomplete Information

The above analysis of partial information shows a non-desirable feature from the bank’s perspective. In this section, we discuss the situation in which the firm has to undertake some costly efforts, ε , to diminish the “noise” acting on the true state of her surplus. However, the bank cannot discriminate between firms which undertake high efforts to disclose their true surplus value from those that do not spend time and costs on this issue. Thus, in addition to having partial information on the state variable, the bank has incomplete information on whether the firm puts a high effort to diminish the signal’s noise. Since a high effort is costly, the firm has no incentive to fully disclose her true surplus state. To remedy this situation, the bank has the possibility to set up a contract with the firm. This contract should be designed along the following lines:

- i) The bank rewards the efforts to disclose the firm’s surplus state by using a compensation scheme or contract. The effort function is assumed to be private information to the firm and, therefore, defines incomplete information for the bank.
- ii) The contract should be incentive compatible for the firm and be at least as good as the next best opportunity.

The first requirement defines a contracting setup where the action of the firm is hidden to the bank (see Grossman and Hart (1983) for the general theory). Since the surplus cannot be observed by the bank, it is reasonable that also the firm’s efforts to disclose the true surplus cannot be observed. This implies that a high value for A_1/B can result either from low efforts or from high efforts, but the latter situation is much more probable. In the second requirement, we add incentive compatibility since we assume that information in the new contract variables is asymmetric between the firm and the bank. We assume quasi-linear utility functions for both the bank and the firm, which are thrice differentiable. For the bank, the formal model with incomplete and partial information reads:

$$(\widetilde{\mathbf{B3}}) \quad : \quad J^B(\hat{S}, t) = \max_{\ell \in \hat{\mathcal{B}}, K \in \mathcal{K}} \mathbb{E}^\varepsilon \left[\int_t^T e^{-\delta(T-s)} \left(p \hat{c}_s \hat{S}_s(\varepsilon) - \kappa \ell_s - K(\varepsilon) \right) ds \middle| \mathcal{G}_t \right]. \quad (11)$$

subject to

$$\varepsilon^*, c_t^* \in \operatorname{argmax}_{c_t \in \hat{\mathcal{C}}, \varepsilon} \mathbb{E}^\varepsilon \left[\int_t^T e^{-\delta(T-s)} (\lambda S_s(\varepsilon) - \omega S_s^2(\varepsilon) + K(\varepsilon)) ds \mid \mathcal{F}_t \right] \quad (12)$$

$$\bar{u} \leq \mathbb{E}^\varepsilon \left[\int_t^T e^{-\delta(T-s)} (\lambda S_s(\varepsilon) - \omega S_s^2(\varepsilon) + K(\varepsilon)) ds \mid \mathcal{F}_t \right], \quad (13)$$

$$dS_t = (\mu_0 + (\mu_1 - (p + \varepsilon))c_t)dt + (\sigma_0 + \sigma_1 c_t)dW_t. \quad (14)$$

Choosing an effort to disclose the surplus state is costly for the firm, but, on the other side, also beneficial. This is reflected by the dynamics of the modified non-observable surplus in condition (14) that contains both the reward $K(\varepsilon)$ through c_t and a cost function for the chosen effort. For simplicity, the cost function is assumed to be the identity function. The reward adjusts the return μ_1 and the cost function adjusts the price p paid for the credit usage. The incentive constraint (12) ensures that the firm always chooses an effort which is in her self-interest. The constraint (13) is the participation or individual-rationality constraint of the firm. It states that the firm accepting a contract is not better off choosing any alternative contract that gives the firm a constant utility \bar{u} .

To formally define such an implementable allocation, we assume that the effort level ε ranges in a closed and compact interval $[\underline{\varepsilon}, \bar{\varepsilon}]$. The bank has the prior cumulative distribution function F^ε with differentiable density f^ε , such that the density is strictly positive for ε . Therefore, the expectation in the bank's objective function given in $(\widetilde{\mathbf{B3}})$ taken under the product measure of the distribution of firm types ε and the surplus distribution.

The program $(\widetilde{\mathbf{B3}})$ generalizes the standard theory of contracts with hidden information⁴ in two respects: First, $(\widetilde{\mathbf{B3}})$ is a dynamic program. Second, the state variable S is not observable for the bank. Instead, there is a noisy signal, from which a best guess on the true state can be extracted. This defines partial information about the state variable.

We next provide a precise model formulation and the corresponding solution concept step-by-step. This is done in *Step I* to *Step VI*.

Step I: The bank estimates the surplus state \hat{S} using the non-linear filter technique presented in Section 3. Repeating the calculations of Section 3, the dynamics of the bank's best guess for the firm's surplus is obtained as

$$d\hat{S}_t = \left(\mu_0 + (\mu_1 - (\varepsilon + p))\hat{c}_t + \rho_t \frac{A_1^2}{B^2} (S_t - \hat{S}_t) \right) dt + \rho_t \frac{A_1}{B} dZ_t, \quad (15)$$

where the solution for ρ_t is given in equation (A.7).

⁴See Mirrless (1971), Grossman and Hart (1983), Fudenberg and Tirole (1993).

Step II: We restrict the agents' optimal decisions by imposing two conditions, $\hat{c}_t = \frac{\hat{\ell}_t}{\hat{S}_t}$ and $\hat{c}_t \hat{S}_t = c_t S_t$. These two conditions allow us to eliminate the decision variable ℓ_t . The first condition is based on the observation that, under incomplete information, it should never pay to offer a higher limit amount than the one calculated by using the best guess on the firm's surplus. The rationale for the second condition is that, given a firm demands credit, the bank cannot decide on c_t and S_t separately due to partial information. The bank only observes $c_t S_t$.

Step III: The contract allocation consisting of the chosen credit demand $c(\varepsilon)$ and the payment $K(\varepsilon)$ have to satisfy the following requirements: The payments resulting from the contract have to be feasible and satisfy the individual rationality constraint. Given the set of feasible allocations, the bank finally chooses the contract payments in the class with the highest expected payoff.

Definition 1. *A decision function $c(\varepsilon)$ is implementable if there exists a payment $K(\varepsilon)$ such that the type-contingent allocation $y(\varepsilon) = (c(\varepsilon), K(\varepsilon))$ satisfies the constraint*

$$u(y(\varepsilon), S(\varepsilon)) \geq u(y(\hat{\varepsilon}), S(\varepsilon)) , \quad \forall (\varepsilon, \hat{\varepsilon}) \in [\underline{\varepsilon}, \bar{\varepsilon}] \times [\underline{\varepsilon}, \bar{\varepsilon}], \quad (16)$$

where the function $u(\cdot)$ is given by:

$$u(y(\varepsilon), S(\varepsilon)) = \int_t^T e^{-\delta(T-s)} (\lambda S_s(\varepsilon) - \omega S_s^2(\varepsilon) + K(\varepsilon)) ds. \quad (17)$$

The constraint in (16) is the standard form for the constraint given in equation (12) of problem $(\widetilde{\mathbf{B3}})$. Basically, it states that the firm optimally consumes the credit corresponding to her effort revealed to the bank. Whenever the revealed effort $\hat{\varepsilon}$ and the true effort ε disagree, the utility for the firm is reduced. Equivalently to equation (17), we will define the bank's utility function as

$$v(y(\varepsilon), S(\varepsilon)) = \int_t^T e^{-\delta(T-s)} \left((p - \kappa) \hat{c}_s(\varepsilon) \hat{S}_s(\varepsilon) - K(\varepsilon) \right) ds. \quad (18)$$

Both functions $u(\cdot), v(\cdot)$, and the decision variable $\hat{c}(\varepsilon)$ are assumed to be sufficiently smooth.

The condition (16) is difficult to handle in dynamic models with a continuous effort space. Therefore, at the price of some extra a-priori assumptions on the functional forms, we introduce some additional necessary conditions.

Proposition 6. Consider the problem $(\widetilde{\mathbf{B3}})$ and (17). The condition

$$\frac{\partial \hat{c}_t(\varepsilon)}{\partial \varepsilon} \frac{\partial^2 u(y(\varepsilon), S(\varepsilon))}{\partial \varepsilon \partial c_t} \geq 0 \quad (19)$$

is sufficient for a piecewise function $\hat{c}_t(\varepsilon)$ to be implementable.

Due to the partial information structure, we have two state variables S and \hat{S} . The usual approach in contract theory to deal with the hidden action ε is not applicable here, since the asymmetric information is related to only one state variable. We therefore assume that the bank is choosing an \hat{S} -optimal policy. The definition of this policy is given below.

Definition 2. An \hat{S} -optimal policy for the bank consists of an implementable allocation $y = (c(\varepsilon), K(\varepsilon))$ and satisfies the individual rationality constraint (13) in problem $(\mathbf{B3})$ where the bank always assumes that \hat{S} is the true state.

Step IV: With the above prerequisites, the \hat{S} -optimal program can be formulated as

$$(\mathbf{B3}) \quad : \quad J^B(\hat{S}, t) = \max_{\hat{c}(\varepsilon), K(\varepsilon), \varepsilon} \mathbb{E}^\varepsilon \left[v(y(\varepsilon), \hat{S}(\varepsilon)) | \mathcal{G}_t \right]. \quad (20)$$

subject to (15) and

$$\mathbb{E}^\varepsilon \left[u(y(\varepsilon), \hat{S}(\varepsilon)) | \mathcal{F}_t \right] \geq \mathbb{E}^\varepsilon \left[u(y(\hat{\varepsilon}), \hat{S}(\varepsilon)) | \mathcal{F}_t \right], \quad \forall (\varepsilon, \hat{\varepsilon}) \in [\underline{\varepsilon}, \bar{\varepsilon}] \times [\underline{\varepsilon}, \bar{\varepsilon}] \quad (21)$$

$$\bar{u} \leq \mathbb{E}^\varepsilon \left[u(y(\varepsilon), \hat{S}(\varepsilon)) | \mathcal{F}_t \right], \quad (22)$$

where the solution for ρ_t is given in equation (A.7) and $\hat{S}_0 = \mathbb{E}^\varepsilon [S_0]$.

Step V: The next step to obtain a solution for (20) is to eliminate the transfer payment $K(\varepsilon)$ in the optimization program.

Proposition 7. Subject to (15), the optimization program (20) is equivalent to

$$J^B(\hat{S}, t) = \max_{\hat{c}(\varepsilon)} \mathbb{E}^\varepsilon \left[\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \left(v_1(c(\varepsilon), \hat{S}(\varepsilon)) + u_1(\hat{c}(\varepsilon), \hat{S}(\varepsilon)) - \frac{1 - F^\varepsilon}{f^\varepsilon} \frac{\partial u_1(\hat{c}(\varepsilon), \hat{S}(\varepsilon))}{\partial \varepsilon} \right) f^\varepsilon d\varepsilon | \mathcal{G}_t \right]$$

where $u_1(\hat{c}(\varepsilon), \hat{S}(\varepsilon)) = u(y(\varepsilon), \hat{S}(\varepsilon)) - \int_t^T e^{-\delta(T-s)} K(\varepsilon) ds$ and $v_1(\hat{c}(\varepsilon), \hat{S}(\varepsilon)) = v(y(\varepsilon), \hat{S}(\varepsilon)) + \int_t^T e^{-\delta(T-s)} K(\varepsilon) ds$. The contract is given by

$$K^*(\varepsilon) = e^{\delta(T-t)} \frac{\partial}{\partial t} \left(u_1(\hat{c}(\varepsilon), S(\varepsilon)) - \int_{\underline{\varepsilon}}^{\varepsilon} \frac{\partial u_1(\hat{c}(\tilde{\varepsilon}), S(\tilde{\varepsilon}))}{\partial \tilde{\varepsilon}} d\tilde{\varepsilon} \right).$$

		time	
		static	dynamic
information	complete/partial	(A)	(A) + (B ₁)
	incomplete/partial	(A) + (C)	(A) + (B ₁) + (C) + (B ₂)

Table 1: Static and dynamic optimization, and different information structures.

Hence, if we know the optimal policy c^* , we can explicitly determine the optimal contract $K^*(\varepsilon)$.

Step VI: The final step is to solve for c^* . We content ourselves with stating the optimality condition,

$$\underbrace{\mathcal{A}_{\hat{S}} J^B(\hat{S}, t)}_{(B_1)} + \underbrace{\frac{\partial J^B(\hat{S}, t)}{\partial \hat{c}} + \frac{\partial J^C(\hat{S}, t)}{\partial \hat{c}}}_{(A)} = \underbrace{\frac{1 - F(\varepsilon)}{f(\varepsilon)} \frac{\partial^2 u_1(\hat{c}(\varepsilon), \hat{S}(\varepsilon))}{\partial \varepsilon \partial \hat{c}}}_{(C)} + \underbrace{\Delta \mathcal{A}_{\hat{S}} J^B(\hat{S}, t)}_{(B_2)},$$

and categorize these components in Table 1. By inspection, the bank faces a tradeoff between maximizing the joint surplus, given by the expression (A), and appropriating the firm's information rent, represented by (C). The term (C) is solely determined by factors depending on the characteristics of the firm. If we were in a static economy, the optimum would be obtained when an increase in the joint surplus equals the expected increase in the company's rent.

For $\varepsilon = \bar{\varepsilon}$, the term (C) is just zero and the extraction of the company's effort is not a concern, i.e., only (A) is maximized. Because $\frac{1-F(\varepsilon)}{f(\varepsilon)} \geq 0$, for all other levels of effort ε , the impact of (C) on the bank's marginal utility function depends on the sign of the cross-derivative $\frac{\partial^2 u_1(\hat{c}(\varepsilon), \hat{S}(\varepsilon))}{\partial \varepsilon \partial \hat{c}}$. Since it would be against economic intuition if $\frac{\partial \hat{c}(\varepsilon)}{\partial \varepsilon} < 0$, we have from equation (19) that the sign of $\frac{\partial^2 u_1(\hat{c}(\varepsilon), \hat{S}(\varepsilon))}{\partial \varepsilon \partial \hat{c}} \geq 0$ is positive, i.e., a higher surplus makes a higher demand for credit more desirable. Therefore, the term (C) is strictly positive for $\varepsilon < \bar{\varepsilon}$.

The effect of the effort on the magnitude of (C) is less clear. As noted in Fudenberg and Tirole (1993) (chapter 7), the optimal decision obtained by ignoring the monotonicity constraint satisfies monotonicity, if we replace the monotonicity condition $\frac{\partial \hat{c}(\varepsilon)}{\partial \varepsilon} > 0$ with the assumption $\frac{\partial^3 u_1(\hat{c}(\varepsilon), \hat{S}(\varepsilon))}{\partial \varepsilon^2 \partial \hat{c}} \leq 0$. Therefore, with the sufficient condition $\frac{\partial^3 u_1(\hat{c}(\varepsilon), \hat{S}(\varepsilon))}{\partial \varepsilon^2 \partial \hat{c}} \leq 0$, the term (C) induces the bank to attract debtors with a low effort ε , if and only if the hazard rate $\frac{f(\varepsilon)}{1-F(\varepsilon)}$ satisfies $\frac{d}{d\varepsilon} \left(\frac{f(\varepsilon)}{1-F(\varepsilon)} \right) \geq 0$. Bagnoli and Bergstrom (1989) show that the latter condition is equivalent to $f(\varepsilon)$ being log-concave on $[\underline{\varepsilon}, \bar{\varepsilon}]$. Log-concavity is fulfilled for distributions such as the uniform, normal, logistic,

and χ^2 -distribution, but in general not, e.g., for the Student's-t distribution. Therefore, monotonicity of the cross-derivative and a log-concave distribution function for ε on $[\underline{\varepsilon}, \bar{\varepsilon}]$ make low efforts more desirable for the bank in order to increase the impact of (C) on her value function. In contrast, a non-monotone hazard rate will prevent the bank to attract firms with low effort. We conclude that, from the bank's perspective, the optimal effort level to reveal the true surplus strongly depends on the properties of the bank's prior distribution of the firm's type (effort). The bank tries to obtain a lower effort level, if the prior distribution is log-concave, and a higher effort level if the hazard rate is decreasing in $[\underline{\varepsilon}, \bar{\varepsilon}]$. The optimal effort level not only depends on the behavior of the quotient $\frac{1-F(\varepsilon)}{f(\varepsilon)}$, but also on its influence on other terms. When we move from a static to a dynamic setup, there are two intertemporal components entering the scene, (B_1) and (B_2) . (B_1) is invariant to the possibility of incomplete information. Contrary to (B_2) , it does not depend on the effort ε . The term (B_2) serves as a correction term for the standard generator $\mathcal{A}_{\hat{S}}$. The generator decreases when the effort level decreases, which reduces the bank's value function. Therefore, (B_2) can be interpreted as a dynamic hedging component against incomplete information on the firm's effort ε , and (B_1) as the dynamic hedging component against the noisy signal.

5 Implication for Credit Risk Management

5.1 Model with Full Information

From the optimal policies derived in Proposition 2, the marginal utility of an additional dollar surplus is always larger for the firm than for the bank. The discount factor and the additional risk of a new investment uniquely determine the difference between the two marginal utilities. When the firm and the bank become more impatient, the increase in the marginal utility from an additional investment turns out to be larger for the bank than for the firm. Similarly, if the risk of new investments decreases to zero, the utility from an additional dollar surplus increases unlimitedly for the bank, whereas for the firm the additional utility tends to zero. Therefore, given full information, banks should finance long-term investments with low risks.

To what extent should this long-term financing grow? The sign of γ_2 gives an unambiguous answer. Given that $\gamma_2 > 0$, the optimal limit policy is not bounded from above. In the other case, once the surplus has reached a critical level, the bank should stop to assign additional credit limits. Under what conditions does the sign of γ_2 turn positive or negative? Separating the surplus dynamics S into a sum $S = S_0 + S_1$, where S_0 denotes the surplus

when $c^* = 0$ and $S_1 = S - S_0$, then the inequality $\gamma_2 > 0$ holds if

$$\mathbb{E}(dS_0) \mathbb{E}(dS_1/c^*) > \delta \sqrt{\text{Var}(dS_0)} \sqrt{\text{Var}(dS_1/c^*)}, \quad (23)$$

given that the firm's additional return on investment is larger than the bank's cost of capital, i.e., $\mu_1 > p$.

The inequality (23) allows us to derive several implications. First, the lower the risk of new investments, the more likely inequality (23) holds. As a consequence, the bank should finance any credit demands. Second, if the bank and the firm are increasingly impatient, then an optimizing bank will bound the total limit usage once the surplus of the firm reached a given known level.

So far, no distinction was made whether a firm asked for one million or one billion dollars. The price of one dollar credit p was assumed to be constant. If we consider a small firm asking for a one million dollar credit and a large firm asking for a one billion dollar credit, then a discount, i.e., $p_1 > p_2$, is a reasonable assumption. Since γ_2 is strictly decreasing in p , the inequality (23) holds more likely for the small than for the large firm. Hence, if we charge a higher price to the small firm, it is rational to bound the total credit limit of the large firm for a known surplus level. Put differently, a large firm cannot expect both a discount and unlimited resources from the bank's side.

5.2 Model with Partial Information

In this model, the bank has not full information about the firms surplus. Instead, the bank observes a noisy signal. The firm can exploit the bank's lack of full information through manipulating the signal and thereby rising the credit limits. Basically, this can be done in two ways. First, the firm can produce a highly noisy signal, since it is in the firm's self-interest if a large amount of noise hides their true firm value or surplus state. Second, the firm may behave in such a way so as to weaken the correlation between the nonobservable surplus and the observable signal. To the extreme, the firm can even try to make the correlation negative.

However, it is not optimal for all firms to behave in this way. Consider a firm whose risk and return properties satisfies the analogue of inequality (23) for the partial information case. Then, the firm has a less pronounced incentive to produce information distortions, since in any case it is optimal for the bank to cover any credit demand. Therefore, only a firm with less favorable risk and return properties will have an incentive to manipulate the signal as described above.

5.3 Model with Partial and Incomplete Information

If the bank faces partial information, the bank will counteract the firm's incentives to produce noisy signals. Since it is difficult to assess the firm's effort to disclose her surplus status, we assume that the bank has only incomplete information about the firm's effort. Therefore, the bank has to work out a clever contracting scheme such that it is optimal for the firm to reveal the true effort. Such a procedure leads to second-best contracts, i.e., the resolution of the incompleteness leads to an optimal decision that is costly and less efficient than the policy with complete information. Therefore, given the possibility of efficiency losses, institutions should decide whether it is worth to work out an incentive compatible contract to meet the firm's credit demand. The alternative would be to simply refuse the credit demand.

If a bank decides to enter into the contract, a key model risk is the prior distribution function used to assess the 'type' of the counterparty. This probability function measures the probability that the firm chooses a low, a medium, or a high effort level. The assessment of these probabilities has far reaching consequences, since it basically determines which types of firms the bank attracts. This attraction will finally determine the credit portfolio's composition.

As an example, we consider a bank that assumes the distribution to be uniform for one client and Student's t distributed for a second one. To simplify the analysis, we consider a static setup. At the optimum, up to constants, the bank's value function J^B and firm's value function J^C satisfy

$$J^B = -J^C + (\bar{\epsilon} - \epsilon) \frac{\partial^2 u_1(\hat{c}(\epsilon), \hat{S}(\epsilon))}{\partial \epsilon \partial \hat{c}}, \quad (24)$$

for the uniform distribution with $\bar{\epsilon}$ the maximum effort level, and

$$J^B = -J^C + \frac{(1 + \epsilon^2) \left(\pi - 2 \arccos\left(\sqrt{\frac{1}{1+\epsilon^2}}\right) \text{sgn}(\bar{\epsilon}) \right)}{2} \frac{\partial^2 u_1(\hat{c}(\epsilon), \hat{S}(\epsilon))}{\partial \epsilon \partial \hat{c}}, \quad (25)$$

for the Student's t distribution with one degree of freedom. For the uniform distribution, since equation (25) is monotone increasing in ϵ , the bank prefers firms with low effort level. For the Student's t distribution, the bank prefers firms with a high effort level. Hence, if the firm has no information at hand (uniform case), different types of counterparties enter the bank's credit portfolio. In the worst case, wrong prior beliefs will lead to an unintended composition of the bank's credit portfolio.

6 Conclusion

The increasing importance of limit management is related to the deterioration of credit markets during recent years, which makes short-term decisions an indispensable tool for adequate risk management. We analyze limit management under different information setups. We start with a model, where both the bank and the firm have complete information about the firm's state variable. Since small variations in the model parameters lead to large differences in the optimal limit policy and since banks often have to assess a credit limit with partial information on the firm's true surplus we are led to consider a partial information model using noisy signals. But a noisy signal for the firm's true surplus introduces some undesirable effects for the bank since incentives of the firm are distorted. However, these incentive distortions can be reduced by implementing an incentive optimal contract for the firm to put some effort into disclosing her true surplus. This, in turn, gives rise to an optimization problem in the presence of incomplete information. Setting up an incentive-compatible contract may turn out to be costly. Furthermore, the success of such contracting strongly depends on the correct assessment of the prior distribution of counterparty qualities.

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Appendix

Proof of Proposition 2 and Proposition 3. For company C , the value function J^C must satisfy $0 = \max_{c_t \in \mathcal{C}} (e^{-\delta(T-t)} (\lambda S_t - \omega S_t^2) + \mathcal{L}J^C - \phi(c_t S_t - \ell_t))$, where ϕ is the Lagrange multiplier and \mathcal{L} is the extended generator of S . Solving for c^* and making the Ansatz $J^C(S, t) = e^{-\delta(T-t)} (k_0 + k_1 S_t + \frac{1}{2} k_2 S_t^2)$, we obtain the parameters k_i as

$$\begin{aligned} k_0 &= \frac{k_1^2 (\mu_1 - p)^2 - 2k_2 \phi \sigma_2^2 \ell_t + 2k_1 k_2 \sigma_1 ((\mu_1 - p) \sigma_0 - \mu_0 \sigma_1)}{2k_2 \delta \sigma_1^2}, \\ k_1 &= k_2 \sigma_1 \frac{(\phi - k_2 (\mu_1 - p)) \sigma_0 + (\lambda + k_2 \mu_0) \sigma_1}{(\phi - k_2 (\mu_1 - p)) (p - \mu_1) - k_2 \delta \sigma_1^2}, \\ k_2 &= \frac{\phi(\mu_1 - p) - \sigma_1^2 \omega + \sigma_1 \sqrt{\delta \phi^2 - 2\phi \omega (\mu_1 - p) + \sigma_1^2 \omega^2}}{(\mu_1 - p)^2 - \delta \sigma_1^2}. \end{aligned}$$

Determining the Lagrange multiplier and substituting into the expression for c^* , we get $c_t^* = \frac{\ell_t}{S_t}$. For the bank, the HJB equation is $0 = \max_{\ell \in \mathcal{B}} (e^{-\delta(T-t)} (p c_t S_t - \kappa \ell_t) + \mathcal{L}J)$. With $J^B(S, t) = e^{-\delta(T-t)} (p - \kappa) (b_0 + b_1 S_t + \frac{1}{2} b_2 S_t^2)$, the coefficients b_i are obtained as

$$b_0 = \frac{(p - \mu_1) (\delta \sigma_0^2 - \mu_0^2) (\mu_1 - p + 2\sqrt{\delta} \sigma_1) - 2\delta \mu_0 \sigma_1^2 (\mu_0 + \sqrt{\delta} \sigma_0)}{2\delta^2 (\mu_1 - p + \sqrt{\delta} \sigma_1)^3}, \quad (\text{A.1})$$

$$b_1 = \sigma_1 \frac{\mu_0 + \sqrt{\delta} \sigma_0}{\sqrt{\delta} (\mu_1 - p + \sqrt{\delta} \sigma_1)^2}, \quad (\text{A.2})$$

$$b_2 = -\frac{1}{2 (\mu_1 - p + \sqrt{\delta} \sigma_1)}. \quad (\text{A.3})$$

Plugging these results into the expression for ℓ_t^* , we obtain γ_1 and γ_2 as claimed in the proposition. Finally, plugging the optimal limit policy into the value function of C , we obtain the parameters k_i as

$$k_0 = -\frac{2k_1 (\gamma_1 (\mu_1 - p) + \mu_0) + k_2 (\sigma_0 + \gamma_1 \sigma_1)^2}{2\delta}, \quad (\text{A.4})$$

$$k_1 = -\frac{\lambda + k_2 (\mu_0 + \gamma_2 \sigma_0 \sigma_1 + \gamma_1 (\mu_1 - p + \gamma_2 \sigma_1^2))}{\delta + \gamma_2 (\mu_1 - p)}, \quad (\text{A.5})$$

$$k_2 = \frac{2\omega}{\delta + \gamma_2 (2(\mu_1 - p) + \gamma_2 \sigma_1^2)}. \quad (\text{A.6})$$

This concludes the proof. □

Proof of Proposition 5. The HJB equation of the company has solution $J^C(S, t) = e^{-\delta(T-t)} (\hat{k}_0 + \hat{k}_1 S_t + \frac{1}{2} \hat{k}_2 S_t^2)$. Evaluating the Lagrange multiplier and plugging it

into the optimal policy, yields $c^* = \ell_t/S_t$. The bank's HJB equation is given by

$$0 = \max_{\ell \in \mathcal{B}(\hat{S})} \left(e^{-\delta(T-t)} \left(p\hat{c}_t\hat{S}_t - \kappa\ell_t \right) + \hat{\mathcal{L}}J^B \right),$$

where $\hat{\mathcal{L}}$ is the extended generator of the filtered process given in (8). From Proposition 4, ρ_t is obtained as

$$\rho_t = \frac{(\sigma_0 + \sigma_1\hat{c}_t)B^2}{A_1^2} \times \frac{\frac{A_1^2}{B^2}\rho_0 - (\sigma_0 + \sigma_1\hat{c}_t) + e^{2(\sigma_0 + \sigma_1\hat{c}_t)\frac{A_1^2}{B^2}t} \left(\frac{A_1^2}{B^2}\rho_0 + (\sigma_0 + \sigma_1\hat{c}_t) \right)}{(\sigma_0 + \sigma_1\hat{c}_t) - \frac{A_1^2}{B^2}\rho_0 + e^{2(\sigma_0 + \sigma_1\hat{c}_t)\frac{A_1^2}{B^2}t} \left(\frac{A_1^2}{B^2}\rho_0 + (\sigma_0 + \sigma_1\hat{c}_t) \right)}.$$

The above equation is non-linear in \hat{c}_t and has no closed-form solution. The parameter \hat{c}_t always appears in ρ_t scaled by σ_1 . We argue that $\hat{c}_t\sigma_1$ is usually small and linearize ρ_t by using a first-order approximation around $\hat{c}_t\sigma_1 = 0$. Then, $\rho_t^{(1)}$ defined by $\rho_t = \rho_t^{(1)} + O\left((\hat{c}_t\sigma_1)^2\right)$ is

$$\begin{aligned} \rho_t^{(1)} &= \frac{\sigma_0 B^2}{A_1^2} \times \frac{\frac{A_1^2}{B^2}\rho_0 \cosh\left[\frac{A_1^2}{B^2}\sigma_0 t\right] + \sigma_0 \sinh\left[\frac{A_1^2}{B^2}\sigma_0 t\right]}{\sigma_0 \cosh\left[\frac{A_1^2}{B^2}\sigma_0 t\right] + \frac{A_1^2}{B^2}\rho_0 \sinh\left[\frac{A_1^2}{B^2}\sigma_0 t\right]} + \frac{\sigma_1 B^2}{A_1^2} \\ &\times \frac{e^{4\frac{A_1^2}{B^2}\sigma_0 t} \left(\frac{A_1^2}{B^2}\rho_0 + \sigma_0\right)^2 - \left(\frac{A_1^2}{B^2}\rho_0 - \sigma_0\frac{A_1^2}{B^2}\right)^2 - 4\frac{A_1^2}{B^2}\sigma_0 e^{2\frac{A_1^2}{B^2}\sigma_0 t} \left(\rho_0 + \frac{A_1^4}{B^4}\rho_0^2 t - \sigma_0^2 t\right)}{\left(\sigma_0 - \frac{A_1^2}{B^2}\rho_0 + e^{2\frac{A_1^2}{B^2}\sigma_0 t} \left(\frac{A_1^2}{B^2}\rho_0 + \sigma_0\right)\right)^2} \hat{c}_t, \\ &=: r_0(t) + r_1(t)\hat{c}_t. \end{aligned} \tag{A.7}$$

Rewrite the HJB equation as

$$\begin{aligned} 0 &= \left(\mu_0 + r_0(t)\frac{A_1^2}{B^2} \left(S_t - \hat{S}_t \right) + \left(\mu_1 - p + r_1(t)\frac{A_1^2}{B^2} \left(S_t - \hat{S}_t \right) \right) \hat{c}_t \right) J_{\hat{S}}^B \\ &+ \frac{A_1^2}{2B^2} (r_0(t) + r_1(t)\hat{c}_t)^2 J_{\hat{S}\hat{S}}^B + p\hat{c}_t\hat{S}_t - \kappa\ell_t + J_t^B, \end{aligned} \tag{A.8}$$

which is hard to solve. We therefore follow a perturbative approach and assume that $\left| \frac{S_t - \hat{S}_t}{\hat{S}_t} \right|$ is small. Expanding in $\left| \frac{S_t - \hat{S}_t}{\hat{S}_t} \right|$ around zero, we obtain

$$0 = p\hat{c}_t\hat{S}_t - \kappa\ell_t + J_t^B + (\mu_0 + (\mu_1 - p)\hat{c}_t) J_{\hat{S}}^B + \frac{A_1^2}{2B^2} (r_0(t) + r_1(t)\hat{c}_t)^2 J_{\hat{S}\hat{S}}^B.$$

Now, determining the J^B function is equivalent to the proof of Proposition 2, but with σ_0 and σ_1 replaced by $r_0(t)\frac{A_1}{B}$ and $r_1(t)\frac{A_1}{B}$, respectively. \square

Proof of Proposition 6. We define $\Phi(\hat{\varepsilon}, \varepsilon) := u(y(\hat{\varepsilon}), S(\varepsilon))$. Maximizing Φ pointwise yields the first-order (FOC) and second-order (SOC) optimality conditions at the optimum $\varepsilon = \hat{\varepsilon}$. For notational convenience, we slightly abuse our notation. No

confusion should occur. Then,

$$\text{FOC} : \frac{\partial \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \hat{\varepsilon}} \Big|_{\varepsilon=\hat{\varepsilon}} = 0 ; \quad \text{SOC} : \frac{\partial^2 \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \hat{\varepsilon}^2} \Big|_{\varepsilon=\hat{\varepsilon}} \leq 0.$$

Differentiating the FOC, we get

$$\frac{\partial}{\partial \hat{\varepsilon}} \Big|_{\varepsilon=\hat{\varepsilon}} \left(\frac{\partial \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \hat{\varepsilon}} \Big|_{\varepsilon=\hat{\varepsilon}} \right) = \frac{\partial^2 \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \hat{\varepsilon}^2} + \frac{\partial^2 \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \hat{\varepsilon} \partial \varepsilon} = 0 \Rightarrow \text{SOC} \Leftrightarrow \frac{\partial^2 \Phi(\hat{\varepsilon}, \varepsilon)}{\partial \hat{\varepsilon} \partial \varepsilon} \geq 0.$$

Using the explicit expressions for $u(\cdot)$, we get after some calculations $\frac{\partial^2 \Phi}{\partial \hat{\varepsilon} \partial \varepsilon} \Big|_{\varepsilon=\hat{\varepsilon}} = \int_t^T ds e^{-\delta(T-s)} \frac{\partial c_s}{\partial \varepsilon} \frac{\partial^2 u(y(\hat{\varepsilon}), S(\varepsilon))}{\partial \varepsilon \partial c_s}$. Thus, $\frac{\partial^2 \Phi}{\partial \hat{\varepsilon} \partial \varepsilon} \geq 0 \Leftrightarrow \frac{\partial c_s}{\partial \varepsilon} \frac{\partial^2 u(y(\hat{\varepsilon}), S(\varepsilon))}{\partial \varepsilon \partial c_s} \geq 0$. \square

Proof of Proposition 7. We define the indirect utility function $\hat{U}(\varepsilon) := \max_{\hat{\varepsilon}} \Phi(\hat{\varepsilon}, \varepsilon)$. Using the Envelope Theorem we get

$$\frac{d\hat{U}(\varepsilon)}{d\varepsilon} = \frac{\partial u(y(\varepsilon), S(\varepsilon))}{\partial \varepsilon} = \frac{\partial u_1(\hat{c}(\varepsilon), S(\varepsilon))}{\partial \varepsilon}.$$

This implies

$$\hat{U}(\varepsilon) = \underline{\hat{U}} + \int_{\underline{\varepsilon}}^{\varepsilon} \frac{\partial u_1(\hat{c}(\tilde{\varepsilon}), S(\tilde{\varepsilon}))}{\partial \tilde{\varepsilon}} d\tilde{\varepsilon}.$$

Since the bank maximizes the utility from the joint surplus minus the agent's utility, we get $K(\varepsilon) = \hat{U}(\varepsilon) - v(y(\varepsilon), S(\varepsilon))$, with $v(\cdot)$ defined in (18). The participation constraint (13) then implies $\underline{\hat{U}} = 0$. Hence,

$$\int_t^T e^{-\delta(T-s)} K(\varepsilon) ds = \int_{\underline{\varepsilon}}^{\varepsilon} \frac{\partial u_1(\hat{c}(\tilde{\varepsilon}), S(\tilde{\varepsilon}))}{\partial \tilde{\varepsilon}} d\tilde{\varepsilon} - u_1(\hat{c}(\varepsilon), S(\varepsilon)).$$

Once we know $c(\varepsilon)$, we know $K(\varepsilon)$. Replacing $K(\varepsilon)$ in the utility function of the bank and carrying out one partial integration, we have proven our claim. \square