Modeling Credit Migration

Credit models are increasingly interested in not just the probability of default, but in what happens to a credit on its way to default. Attention is being focused on the probability of moving from one credit level, or rating, to another. One convenient way of expressing this information is through a transition matrix. The primary source for these probabilities has been the rating agencies. As an example, Figure 1 contains the historical frequency of annual transitions based on S&P observations from 1981 to 1998.

There are, however, a number of difficulties involved in applying historical matrices in practice:

- In spite of the large number of observations, some measurements have low statistical significance, particularly the investment grade default frequencies. For example, the BBB default statistic is 0.24% per year, its value at the 95% confidence interval is over 200% greater.\(^2\)
- The observations are based on seven major rating categories. Many applications require a finer granularity in credit levels.
- Historical observations do not reflect to the current credit environment.
- Historical observations are primarily based in the U.S. and are inappropriate for emerging markets.
- History only provides an assessment of real probabilities. Pricing models need probabilities which are adjusted for risk (risk neutral) and are consistent with observed prices in the market.

This article discusses a method for modeling credit migration and default probability. The goal in developing this model was to address the issues discussed above using a simple framework with broad pricing and risk management application. A few highlights:

- Rather than being confined to historical rating transitions, the data set is expanded to include historical cumulative default statistics as well as current credit spreads. As will be shown,

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1 This article originally appeared in RISK, February 2000.
2 The methodology for this assessment closely resembles that described in Kealhofer, Kwok and Weng [1998]. This assumes a 35% asset cross-correlation, which is close to average for BBB.

### S&P: N.R. Adjusted Average One-Year Transition Rates

<table>
<thead>
<tr>
<th>INITIAL RATING</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>91.93%</td>
<td>7.46%</td>
<td>0.48%</td>
<td>0.08%</td>
<td>0.04%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>AA</td>
<td>0.64%</td>
<td>91.81%</td>
<td>6.76%</td>
<td>0.60%</td>
<td>0.06%</td>
<td>0.12%</td>
<td>0.03%</td>
<td>0.00%</td>
</tr>
<tr>
<td>A</td>
<td>0.07%</td>
<td>2.27%</td>
<td>91.68%</td>
<td>5.12%</td>
<td>0.56%</td>
<td>0.25%</td>
<td>0.01%</td>
<td>0.04%</td>
</tr>
<tr>
<td>BBB</td>
<td>0.04%</td>
<td>0.27%</td>
<td>5.56%</td>
<td>87.87%</td>
<td>4.83%</td>
<td>1.02%</td>
<td>0.17%</td>
<td>0.24%</td>
</tr>
<tr>
<td>BB</td>
<td>0.04%</td>
<td>0.10%</td>
<td>0.61%</td>
<td>7.75%</td>
<td>81.48%</td>
<td>7.90%</td>
<td>1.11%</td>
<td>1.01%</td>
</tr>
<tr>
<td>B</td>
<td>0.00%</td>
<td>0.10%</td>
<td>0.28%</td>
<td>0.46%</td>
<td>6.95%</td>
<td>82.80%</td>
<td>3.96%</td>
<td>5.45%</td>
</tr>
<tr>
<td>CCC</td>
<td>0.19%</td>
<td>0.00%</td>
<td>0.37%</td>
<td>0.75%</td>
<td>2.43%</td>
<td>12.13%</td>
<td>60.44%</td>
<td>23.69%</td>
</tr>
</tbody>
</table>
any one of these three sources is inadequate to paint a reliable picture of credit transitions. In aggregate, however, they provide a more robust result.

- Integrating spreads into the analysis requires a means of translating between real and risk neutral probabilities. The approach used here is based on the capital asset pricing model (CAPM) and derivatives pricing. The end result will be two sets of transition probabilities, one “real” and one risk neutral.
- The number of credit levels is expanded to a much larger set, only limited by computational power.
- The process for credit migration is specified in six parameters. An additional seven are used to map the process to the standard rating categories. This represents a large reduction in the number of free variables used in prior work³.
- The model can be used for a wide range of risk management and pricing applications. In particular, it lends itself to portfolio applications by providing a means for determining loss and return distributions.

This work extends the reduced form framework of Jarrow, Lando, and Turnbull [1997] in several ways. In particular, the number of free parameters is reduced by the introduction of a jump-diffusion process, additional data sources are used to bolster confidence in the results, and the model uses a new approach for risk neutral pricing.

The next section describes the credit process and steps through the procedure for calibrating this process to historical observations. Subsequent sections extend the technique to risk neutral space and add spreads to the calibration procedure. Fit results are described throughout.

The Process - The basic underlying or state variable in this model is the one period default probability.⁴ This default rate undergoes changes in accordance with a probability distribution. For credit markets, distributions are highly skewed. For example, the upside of a loan or bond is limited whereas the downside has a significant tail due to defaults. Jump-Diffusion processes are well suited to the asymmetric, fat-tailed distributions of credit. Therefore, the default rate is modeled here as a logarithmic jump-diffusion process.

The rows of an historical transition matrix are not dramatically different from one another, suggesting that the same process might successfully characterize each row. There is one exception to this statement - higher rated credits have a greater downgrade likelihood than lower rated credits. Evidence of this “mean reversion” effect can be observed in the S&P matrix, where a single-A is twice as likely to downgrade as to upgrade, in contrast to the reverse behavior for a single-B⁵. For this reason the model includes a parameter for mean reversion.

³ Examples include Jarrow, Lando, Turnbull [1997] and Arvanitis [1999].
⁴ A one year period is used in the examples here. In theory any time interval can be used. With a few modifications, the process can be adapted to a continuous-time framework as in Jarrow, Lando and Turnbull [1997].
⁵ Downgrade here does not include default.
The continuous credit process is described in Figure 2. The following parameters characterize this process:

- The diffusion volatility, $\sigma_0$.
- The jump probability density, $\lambda$.
- The expected jump size, $k$.
- The standard deviation of jump size, $\Delta k$.

Two parameters determine the drift as a function of the state:

- The state in which the diffusion drift is zero, $\bar{x}$.
- The mean reversion parameter, $\beta$.

The Jump-Diffusion process, conditional on no default:

$$dx = \beta(\bar{x} - x)dt + \sigma_d dz + Ydq$$

where $x$ is the log of the instantaneous default rate, $dz$ is standard Brownian motion and $dq$ is the jump event.

$$dq = \begin{cases} 1 & \text{with } p = \lambda \ dt \\ 0 & \text{with } p = 1 - \lambda \ dt \end{cases}$$

In the event of a jump, $Y$ is normally distributed with mean $k$ and standard deviation $\Delta k$.

**Conversion to discrete time** - The probability of $n$ jumps occurring over the period is determined from a Poisson likelihood.

$$P_n(\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}$$

Let $x_0$ be the log of the one-period default rate at the start of the period and $x$ its value at the end of the period. If $n$ jumps have occurred, the conditional distribution for $x$ is normal with mean:

$$\sigma_n = \sqrt{\sigma_0^2 + n\Delta_k^2}$$

Figure 2: Details on the jump diffusion process in default probability.
Each discrete state corresponds to an interval on the continuous axis, as illustrated in Figure 2 by the tick marks on the x-axis. The probability of the transition to a discrete state is given by the probability of landing in the corresponding interval. For the results in this article, the space was divided into 40 discrete states, although in practice a larger number could be used.

Except for its drift, the distribution of credit migration is the same for all initial credit levels. Due to mean reversion, high rated firms expect an increase in default rate while lower rated firms expect a decrease.

**Fitting the Data** - Three sources of data will be used to fit the model: annual historical rating transitions, cumulative defaults and credit spreads. As a first step, it is useful to fit to the historical data without the credit spreads. Later, after the risk neutral adjustment is introduced, credit spreads will be added to the optimization.

For calibrating to S&P data, the seven standard rating categories are mapped to default probabilities. Although this mapping could be done by judgement, here additional fit parameters are introduced for this purpose. These mapping parameters represent the bounds for the default probabilities of each rating category. For example, the BBB rating category might correspond to default probabilities falling between 0.15% and 0.70%.

In total, six fit parameters characterize the process and seven parameters map the bins to the standard S&P rating categories. The optimum selection of these thirteen free variables occurs by minimizing the deviation of the model predictions from the input data. The deviation is quantified by the objective function,

\[
Objective\ Function = \sum_k w_k (data_k - model_k)^2
\]

where \(data_k\) represents the \(k^{th}\) data value (whether it is a transition probability, a default probability or a spread) and \(model_k\) represents the corresponding theoretical value predicted by the model. The weights, \(w_k\), are directly related to the confidence level of the \(k^{th}\) data value.

In practice the selection of \(w_k\) requires judgment. For example, many of the one-year transitions are small numbers with low statistical significance. Therefore, among the transition weights, a strong emphasis will be placed on the tri-diagonal elements (i.e., no rating change or a single upgrade or downgrade). In addition, the weights can be used to place relative emphasis on the data sources - one-year transitions, multi-year defaults, and credit spreads.

**Results: Historical Data Only** - This section contains the results of an optimization to historical data from S&P [1999]. The one-year transition results are given in Figure 3, and cumulative defaults are listed in figure 4.
Figure 3: Transition probability results from fitting to S&P one-year transitions as well as multi-year cumulative default probabilities.

<table>
<thead>
<tr>
<th>RATING</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>92.97%</td>
<td>3.34%</td>
<td>2.67%</td>
<td>0.91%</td>
<td>0.09%</td>
<td>0.01%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>AA</td>
<td>2.60%</td>
<td>89.20%</td>
<td>4.71%</td>
<td>2.97%</td>
<td>0.46%</td>
<td>0.04%</td>
<td>0.01%</td>
<td>0.01%</td>
</tr>
<tr>
<td>A</td>
<td>0.70%</td>
<td>2.25%</td>
<td>89.77%</td>
<td>5.33%</td>
<td>1.69%</td>
<td>0.19%</td>
<td>0.03%</td>
<td>0.04%</td>
</tr>
<tr>
<td>BBB</td>
<td>0.07%</td>
<td>0.28%</td>
<td>2.47%</td>
<td>90.75%</td>
<td>4.74%</td>
<td>1.24%</td>
<td>0.23%</td>
<td>0.20%</td>
</tr>
<tr>
<td>BB</td>
<td>0.00%</td>
<td>0.01%</td>
<td>0.16%</td>
<td>4.56%</td>
<td>87.79%</td>
<td>4.21%</td>
<td>1.95%</td>
<td>1.33%</td>
</tr>
<tr>
<td>B</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.01%</td>
<td>0.21%</td>
<td>9.05%</td>
<td>78.51%</td>
<td>6.00%</td>
<td>6.23%</td>
</tr>
<tr>
<td>CCC</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.02%</td>
<td>0.38%</td>
<td>15.28%</td>
<td>64.19%</td>
<td>20.14%</td>
</tr>
</tbody>
</table>

Figure 4: Cumulative default probabilities: the first column lists the S&P data, the second shows results of the fit to historical data, the third and fourth columns show results which came from fits that used spreads along with historical data to calibrate the model. The July '99 spreads represent a “normal” market period, the October '98 spreads represent a stressed period.

If the process were to be fitted to the one-year data alone, a set of fit parameters can be found which faithfully reproduces the S&P transition matrix. However, the one-year transitions and the multi-year defaults can not be perfectly fit simultaneously because these two data sets are inconsistent with each other.

If the historical process is Markovian, a one-year transition matrix already implies future multi-year default levels (as the T-year default probabilities are obtained by raising the one-year matrix to the Tth power). Figure 5 is a comparison for a Double-B credit. The cumulative defaults implied from the S&P one-year matrix are shown by the dotted line. Their actual cumulative default experience are shown by the solid line. The one-year matrix understates their cumulative observations. The model results fall between these two, representing a compromise between the fit to the default probabilities on the one hand and the transition probabilities on the other.

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Because of the discreteness of the rating categories the transitions can not be truly Markovian. This effect, however, is fairly small and is not sufficient to explain the discrepancy described in this paragraph.
Figure 5: Contrasts of cumulative default probabilities as a function of maturity for a Double-B credit: S&P's actual observations (solid line), the result of raising the one-year S&P matrix to the Tth power (dotted line), the results of the model (dashed line).

One explanation for such a data discrepancy is simple statistical error. This would not warrant any adjustment to the weightings. However, the weights can be adjusted if the user has greater confidence in one source over the other. By placing equal aggregate weight on transitions and cumulative defaults the fit here can be considered a well-behaved Markovian version of all of the S&P historical observations.

The Risk Neutral Adjustment: Including Spread Data – In this section market spreads are added to the optimization. That is, the process is fit to market prices in addition to historical defaults and transitions. Thus far the model specifies real probabilities. Two additional parameters are needed to get from real probabilities to market prices: a recovery rate, RR, and a market risk premium, θ. The recovery rate is the fraction of notional assumed to be paid upon default\(^7\). The market risk premium is the excess expected return on the market as a fraction of the standard deviation of its return, commonly referred to as the Sharpe Ratio. It will be crucial in converting real probabilities into risk-neutral probabilities.

The pricing of credit (a bond or loan) requires risk-neutral default probabilities and a recovery rate. The details are highlighted in Figure 6.

\(^7\) In theory, rating specific (or even name specific) recovery rates could be used. For this article, a uniform recovery rate of 50% is used across all credits.
Risk neutral default probabilities are obtained through a two step process. First the real one-year transition matrix is transformed to a risk-neutral equivalent. Then this one-period matrix is raised to successive powers to determine the cumulative probabilities.

The risk-neutral transformation is accomplished in a structural framework, where all transitions from one credit level to another are driven by the assets of a firm. In this framework, a default occurs when the firm’s assets drop below some level. The risk-neutral probability distribution is the real distribution translated by the risk premium. This approach is based on CAPM and has its origins in the seminal Black-Scholes[1973] paper. Details are highlighted in Figure 7.

Figure 6: The price of a loan or bond.

For spread measurements in this article, agencies were used for the riskless yield instead of treasuries whose liquidity and tax advantages have distorting effects.

\[ \text{Price}(C,T) = \sum_{i=1}^{T} Z_i \times \left[ C \times (1 - p_i^{RN}) + RR \times (p_i^{RN} - p_{i-1}^{RN}) \right] + Z_T (1 - p_T^{RN}) \]

where \( C \) is the coupon, \( T \), the time to maturity, \( Z_i \), the present value of the riskless zero coupon price for maturity \( i \), and \( p_i \) the risk-neutral cumulative default probability for maturity \( i \). Defaults are implicitly assumed to be uncorrelated with interest rates. The par coupon is derived from setting the price to par and solving for the coupon. The theoretical spread is then the excess of the risky par coupon over the riskless par coupon.

8 More generally, this can be viewed as assets relative to liabilities.
9 Black and Scholes present an alternative derivation of their formula based on CAPM. The implications for risk-neutral probabilities are equivalent to the shift adjustment described here.
CAPM is used for the risk-neutral translation. Asset returns are assumed to be normally distributed and are measured in units of standard deviations from the mean*. The risk premium is proportional to the correlation with the market, $\rho$, and is given by $\theta \rho$. The risk-neutral probability distribution for these assets is the true distribution translated by its risk premium. Thus, the risk neutral probability ($p_{RN}$) of one period return ($R$) falling below some barrier ($b$) is given by the cumulative Normal ($N$) of the shifted barrier ($b+\rho \theta$).

$$p_{RN}(R < b) = N(b + \rho \theta)$$

For example, suppose a firm has a 2% probability of default and a risk premium of $\theta \rho = .35$. Default will occur if return is below $N^{-1}(0.02) = -2.05$, where $N^{-1}$ is the inverse of the standard cumulative normal distribution. The risk-neutral probability of default is then $p_{RN}(R < -2.05) = N(-1.7) = 4.4\%$.

For the transition probabilities, let $q_{ij}$ represent the probability that a credit at level $i$ at the beginning of the period is at level $j$ or below at the end of the period ($j=0$ represents default). The risk-neutral transitions are given by

$$p_{ij}^{RN} = q_{ij}^{RN} - q_{ij-1}^{RN}$$

where from above,

$$q_{ij}^{RN} = N[N^{-1}(q_{ij}^{real}) + \rho \theta],$$

and $q_{ij}^{real}$ is the real transition probability from state $i$ to state $j$ or below.

* Although the asset distribution assumption is different from the jump-diffusion used for credit migration, it results in only limited pricing distortion.

**Figure 7: The framework for translating to risk neutral space.**

This approach to risk contrasts with previous work. For example, Jarrow, Lando and Turnbull [1997] make the adjustment to risk-neutral probabilities by increasing the variance of the real probability distribution. As a result, all risk-neutral off-diagonal elements are greater than their real counterparts, including the upgrade probabilities. This is equivalent to the risk-neutral probability of a positive return being greater than its true probability, which goes against intuition. By using a shift in distribution as opposed to an increase in the variance, the CAPM approach avoids this anomaly.

This approach also has the advantage of leading directly to multi-name applications. For example, joint default can be viewed as the event of asset returns for each name being below their respective default levels. Its probability is given by a bivariate normal distribution.

**The role of Risk Premium** - The market risk premium can take on two different roles. It could be used as another fit parameter, allowing more flexibility in fitting both history and current spread levels. Alternatively, it can be fixed at some level (e.g., historical average), and the other parameters can be optimized on that basis. As will be shown, spreads will have more leverage.
upon the implied credit process in the latter approach. The right approach depends upon the application.

- For pricing, where the fit to spreads is primary, the model should solve for the risk premium.
- For risk management, the goal is not fitting to spreads, per se, but in making the best characterization of the true credit environment. Fixing the risk premium in this case allows spreads to influence this characterization.

The role of spreads- Although spreads are important for risk management applications and critical for pricing, they are by themselves insufficient. Were the model to fit purely to spreads we would find that extremely wide ranges of parameters provide equally good fits to the data. These, in turn, would imply very different loss distributions, with different implications for pricing and risk management. The historical data are needed here to lend stability to the solution and the shape of the distribution.

Note on spreads: Corporate spreads are commonly measured relative to government treasury bonds. Alternatively, we have chosen to measure them relative to US agencies because a significant portion of the spread over treasuries is not credit related. This portion has more to do with liquidity and tax advantages, and can represent a large fraction of the spread, particularly for investment grades.

Spread Results – The previous historical fit has already provided a process. This process, combined with a given risk premium implies a set of theoretical spreads. To the extent that the market spreads differ from these predictions, the market is providing new information. The optimizer will seek a compromise between current spreads and history.

In the following two examples, the model is fitted to spread data as well as historical data. Spreads are taken from October 1998 and July 1999, and are represented by the symbols in Figures 10 and 11. Figure 8 details the optimization assumptions.

<table>
<thead>
<tr>
<th>Spread Fit Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recovery Rate: RR = 50%</td>
</tr>
<tr>
<td>Risk premium: θ = 0.7</td>
</tr>
<tr>
<td>Spreads: excess of Merrill Lynch corporate bond yields over US Agencies*</td>
</tr>
<tr>
<td>Equal aggregate weights on spreads, 1yr transitions &amp; cumulative defaults</td>
</tr>
<tr>
<td>Asset correlations with market range from AAA=65% to B=40% (based on KMV)</td>
</tr>
</tbody>
</table>

* Agencies are used for the riskless yield instead of government bonds whose liquidity and tax advantages have distorting effects.

Figure 8: Optimization assumptions.
July 1999 represents a period when the market was reasonably calm. On average, the model is able to characterize all three data sources well. The model spreads (curves in Figure 10) are close to the data values and the one-year transitions (Figure 9) and multi-year defaults (Figure 4) are not far off their historical values. Some deviations do exist. For example, the model results suggest that the market is not pricing in the Single-B default risk suggested by history - the Single-B spread curve overshoots the data point, while the Single-B cumulative defaults undershoot the historical observations. The opposite is true for the short-term Single-A results.

October 98 represents a large deviation between spreads and history across all ratings. During this period spreads blew out by almost 40% over fears of a pending global credit crisis. The model reacts by increasing default probabilities, as can be seen in Figure 5. The resulting theoretical spreads (Figure 12) are below the market.
Had the model been allowed to fit for the October risk premium, the changes in spread levels would have been reflected in a large increase in the premium, but they would have much less of an effect on the default probabilities that gauge the true riskiness of the credit environment. A fixed risk premium together with an equal weighting scheme (on spreads versus history) will produce results that are sensitive to current conditions, but are dampened to market over-reaction.

It is worth noting the choice of market risk premium, or Sharpe Ratio used in these fits. We measured premiums on credit risk over various time periods, with different sources of spreads and consistently found values between 0.7 and 0.8 existing in normal market environments. A premium of 0.7 is high compared to the equity or interest rate market, where values around 0.4 are more typical. Perhaps credit premiums are high due to the huge skew of credit risk compared to that of market risk.

**Conclusion: Lessons learned at J.P. Morgan** - This model is used at J.P. Morgan primarily to determine loss distributions for risk management and pricing and to provide default rates that are sensitive to current credit conditions. Market spreads are assumed to be the best indicators of concurrent loss probability. Conditioning probabilities is important because default likelihoods change significantly through the credit cycle.

The model has been applied to the following situations:
- **Allowance for credit losses** - to determine the cumulative default probabilities used to project expected future losses.
- **Economic capital requirement** - to determine the transition matrix used to generate the credit portfolio’s loss distribution.
- **Stress testing** - to set credit spreads for use in firm-wide stress testing
- **Pricing structured deals** - to determine a risk neutral transition matrix for generating portfolio loss distribution and timing.

Several lessons have been learned through applications of the model.
- The model’s output is stable and reasonable, over various spread environments (i.e., stressed and unstressed) and sources of historical default information (i.e., US versus emerging markets).
- Combining historical data with current market spreads results in more reasonable and credible estimates of transition and default likelihood than either source alone.
- Where credit risks are being hedged, this model improves the consistency and accuracy of hedges.
- It’s beneficial from a management viewpoint to have a set of consistent default probabilities that can be used across various applications, such as credit derivatives pricing, the allowance for credit losses and the economic capital for the credit portfolio. This increases management confidence in the credibility of complex credit models.
- Management judgment is still important, particularly the relative weighting given to historical default information versus market credit spreads.

Based on these observations, it’s expected the model will be used even more broadly in the future.


