ECONOMIC CAPITAL ASSESSMENT VIA COPULAS: AGGREGATION AND ALLOCATION OF DIFFERENT RISK TYPES[1]

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EXECUTIVE SUMMARY

The most common approach in estimating the inter-risk diversification effect is to aggregate the stand alone economic capital for each risk type through a variance/covariance Markowitz approach. In this paper we propose an alternative framework, still "ex-post" in the sense that marginal models for the loss distribution of each risk are independently developed and then merged to a joint distribution, but based on the simulation of a t-copula dependence structure; the Kendall's τ*s measure is used to correlate the risks. No assumption is then required on the marginal distributions, which depend in fact on the specific models used to describe each risk. The method is applied to calculate the economic capital for an hypothetical portfolio of risks typically faced by a commercial bank. A Window Conditional Expectation approach for allocating back the diversified capital has also been experimented.*

KEYWORDS: copula, risk aggregation, stressed correlations, risk contribution, economic capital, capital allocation, expected shortfall, window conditional expectation, Basel II, Pillar II.

1 INTRODUCTION

To determine the total group-wide economic capital for a bank, standalone economic capital values calculated for each risk type must be aggregated. A simple approach to this problem of aggregation would be to compute the simple sum of the individual stand alone economic capital values for each risk type; however, this naïve approach overestimates the capital that is required at the aggregate level, as it is highly inplausible that the worst-case scenarios for each risk type will materialise simultaneously. Empirical evidence suggests that the various risk types diversify against one another to some degree: this translates into a total worst-case economic capital that is smaller than the simple sum.

The techniques proposed in the literature for combining marginal risk distributions into an overall distribution can be divided into base-level and top-level cathegories. In the base-level aggregation approach, the idea is to identify risk factors which have most influence on the different risk types and develop a simultaneous model, including a description of the common dependency structure of the risk factors. In the top-level aggregation approaches, marginal models for the loss distributions are independently developed for each risk type, and then merged into a joint distribution.

The most common and simple top-level method used for estimating the diversification effect, i.e. the difference between the simple sum and the true aggregated economic capital, is the variance/covariance framework *à la* Markowitz, through a matrix of correlation coefficients that reflect the degree of linear dependency between pairs of random variables (in this case the random variables considered are the stand-alone losses for each of the different risk types). The underlying assumption is that the potential losses resulting from exposure to a risk driver, and as a consequence the overall potential losses, share the same quantiles (this is for instance a feature of normal distributions). The entire economic capital can thus be determined analytically by a function of the correlation coefficients and the separate risk capitals.

This closed-form approach can be easily implemented, but it suffers from the same flawed normality assumption admittedly present in many financial models. Empirical evidence suggests that the normality assumption may not hold as potential losses show fatter tails than the normal distribution. If we take into account the leptokurtic nature of the empirically observed loss distributions, the above analytical solution is not appropriate, and one must resort to simulation – a method that typically involves developing a relatively complex model of the joint probability distribution for the different risk drivers.

An alternative and theoretically more robust method for calculating diversification benefits under extreme conditions is to use copulas. Copula in fact provide a way of isolating the marginal behaviour of individual risk factors from the description of their dependence structure. This approach allows for a rigorous modelling of the dependencies with arbitrary margins and facilitates a more accurate description of extreme outcomes.

The simulation of a copula dependence structure (for instance the t-copula, adopted in this paper) is best suited to aggregate different risk types which are usually faced by banks. The shape of the marginal distributions of risks in this context is not restricted to a specific form but can assume whichever shape best represents the behaviour of the risk factors both at the analytical or at the empirical level.

The copula approach proposed here is still a top-level one. From a theoretical point of view, a system of risk aggregation that takes simultaneously into account how sensitive the recognised risk types are to the various risk drivers seems in fact preferable, but in practice it is more realistic to determine the sensitivity to risk drivers within each risk type, thus preserving an internal capital measure for the single risks (credit, market, ALM, operational, business etc.).

Another problem of the variance/covariance framework is that the linear correlation matrix is typically unable to fully capture the dependency structure between the various risk types, as it is only a measure of linear dependency. This generally underestimates correlation coefficients in extreme cases and consequently overestimates the diversification benefit in the calculation of the economic capital.

The use of linear correlation is sound only for elliptical models[2], which is not the case, even with the t dependence structure, if the marginal are not themselves t-distributions. Apart from elliptical models, rank correlations, which are simple scalar measures of dependence that depend only on the copula of a joint distribution and not on the marginal distributions, are best suited to calibrate the copula to empirical data. They are in fact invariant, in the sense that the simulated numbers exactly replicate the same rank correlations which are used to generate them, which is not generally the case for linear correlations. In addition, rank correlations are more efficient and robust as not influenced by the historical data sample distribution used for their estimation.

Once the aggregate economic capital has been calculated, it is reallocated to the different risk types according to their contribution to the Window Conditional Expectation, defined as the Expected Loss computed in a limited pre-determined interval around the percentile defined for capital computation. The allocation of the VaR based economic capital under this approach appears to be more reasonable than using the Expected Shortfall, due to the inhomogeneity of the risk factor distributions. One further advantage of the Window Conditional Expectation approach is that the window of loss values considered is centered on the economic capital value being estimated.

[2] Broadly speaking, they are called in this way because the contour line of jont distribution has elliptical shape.

CHART 1 Risk aggregation process.

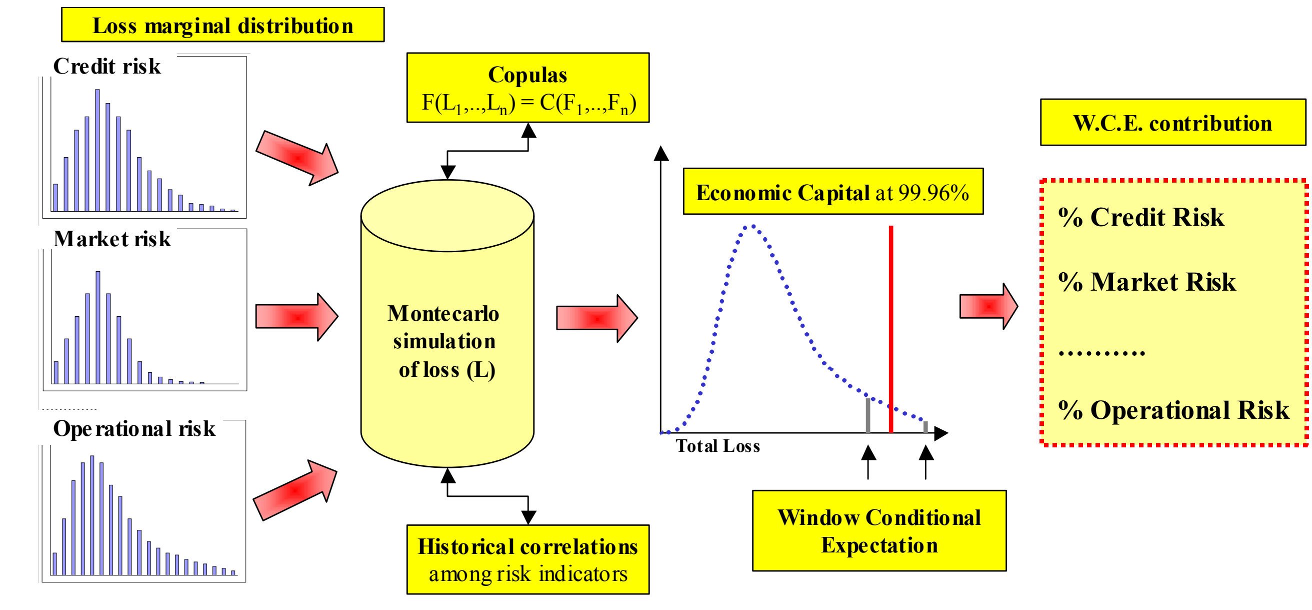


Chart 1 represents the main steps of the overall economic capital computation, outlining both the aggregation of different risk types as well as reallocation to the various risk sources.

The following part of this document is organised as follows: Section 2 outlines the main methodological issues concerning the chosen model for aggregation and reallocation of economic capital; Section 3 describes the results of an application of this methodology to a portfolio of typical bank's risks; and Section 4 provides some concluding remarks on the application developed and the results produced.

2 THE METHODOLOGY

2.1 The variance/covariance approach and its drawbacks

The standard methodology for estimating aggregate economic capital is the variance/covariance formula *à la* Markowitz:

$$VaR_p(\alpha) = \sqrt{\sum_{i,j} \rho_{i,j} VaR_i(\alpha)\, VaR_j(\alpha)} \quad (1)$$

where $VaR_i(\alpha) = \omega_i * \sigma_i * F^{-1}(\alpha)$ is the stand alone VaR for risk i, ω_i the related exposure, σ_i its volatility and $F^{-1}(\alpha)$ the α^{th} quantile of the standardized portfolio losses. The ρ's are the linear correlations between the risks pairs.

Through this formula, the diversified economic capital can be reallocated to the different risk sources according to their marginal contribution (or component VaR):

$$CVaR_i(\alpha) = \frac{VaR_i(\alpha)\sum_{i,j}\rho_{i,j}\, VaR_j(\alpha)}{\sqrt{\sum_{i,j}\rho_{i,j} VaR_i(\alpha)\, VaR_j(\alpha)}} \tag{2}$$

where $CVaR_i(\alpha)$ is the diversified economic capital relative to risk component i, while $VaR_i(\alpha)$ is its stand-alone capital.

As noticed, this technique relies on the implicit constrain that the quantiles of the portfolio are the same as the quantiles of the marginal distributions, which is true for instances where all distributions are normal, but not when some of the risks have fat tailed (or non-normal) distributions. If we substitute stand alone VaR in (1) and then take into account in some way of tails, estimation of true volatility would be biased, leading to an under/over estimation of economic capital depending on the shape of the marginal distributions considered. Furthermore, the linear correlations do not fully capture the entire dependence structure apart from the case of elliptic multivariate distributions (like the Gaussian one): once again, in case of fat tailed distributions, this leads to a misleading estimation of diversification benefits.

2.2 Risks convolution through copula functions

The method we have chosen to apply to calculate the aggregate total economic capital is the copula approach, which allows us to use complex marginal models with a variety of possible dependence structures.

From a mathematical point of view, a copula can be defined as a cumulative n-dimensional density function, C, with standard uniform marginal distributions. Hence C is a mapping C: $[0,1]^n \rightarrow [0,1]$, of the unit hypercube into the unit interval. The most common notation is $C(u_1,\ldots,u_n)$, with (u_i) being the [0,1] probabilities of the marginal cumulative distributions. The following three properties hold:

(1) $C(u_1,\ldots,u_n)$ is increasing in each component u_i;

(2) $C(1,\ldots 1,u_i,1,\ldots,1) = u_i$ is the marginal function of C_i for each $u_i \in [0,1]$;

(3) C is always non-negative.

The importance of copulas in the study of multivariate distribution functions is summarized by Sklar's theorem (1959), which shows that all continuous multivariate distribution functions contain a unique copula and that copulas may be used in conjunction with univariate distribution functions to construct multivariate distribution functions,

$$F(x_1,\ldots x_n) = C(F_1(x_1),\ldots,F_n(x_n)), \tag{3}$$

where F and F_i are respectively joint and marginal distributions with real domain x_i. A useful corollary derived from uniqueness has the representation

$$C(u_1,...u_n) = F\left(F_1^{-1}(u_1),...,F_n^{-1}(u_n)\right) \quad , \tag{4}$$

which yields to the definition of *implicit* copulas, extracted from well-known multivariate distributions. Typical examples are Gaussian and *t* copulas:

$$C_\rho^{Ga} = \phi_\rho\left(\phi^{-1}(u_1),...,\phi^{-1}(u_n)\right) \tag{5}$$

$$C_{v,\rho}^{t} = t_{v,\rho}\left(t_v^{-1}(u_1),...,t_v^{-1}(u_n)\right) \tag{6}$$

with ϕ and t_v equal to Gaussian and t (with ν degrees of freedom) distributions, and where ρ is the correlation matrix.

In the present application, using the implicit copulas mentioned above for the dependence structure and combining them with arbitrary margins, we build non-elliptical models that can be called respectively meta-Gaussian and meta-t:

$$F^{meta-Ga}(x_1,...x_n) = C_\rho^{Ga}\left(F_1(x_1),...,F_n(x_n)\right) \tag{7}$$

$$F^{meta-t}(x_1,...x_n) = C_{v,\rho}^{t}\left(F_1(x_1),...,F_n(x_n)\right) \tag{8}$$

where the $x_1, \ldots, x_n$ are in this case losses due to different types of risk and the $F_1,\ldots,Fn$ are arbitrary marginal functions (here in part normal and in part empirical).

A comparison between the results obtained through the two dependence structures is presented in the next section. The *t*-copula is chosen because it is best suited for modelling correlated extreme events: it can in fact be demonstrated that even uncorrelated multivariate t-distributed random variables are not independent.

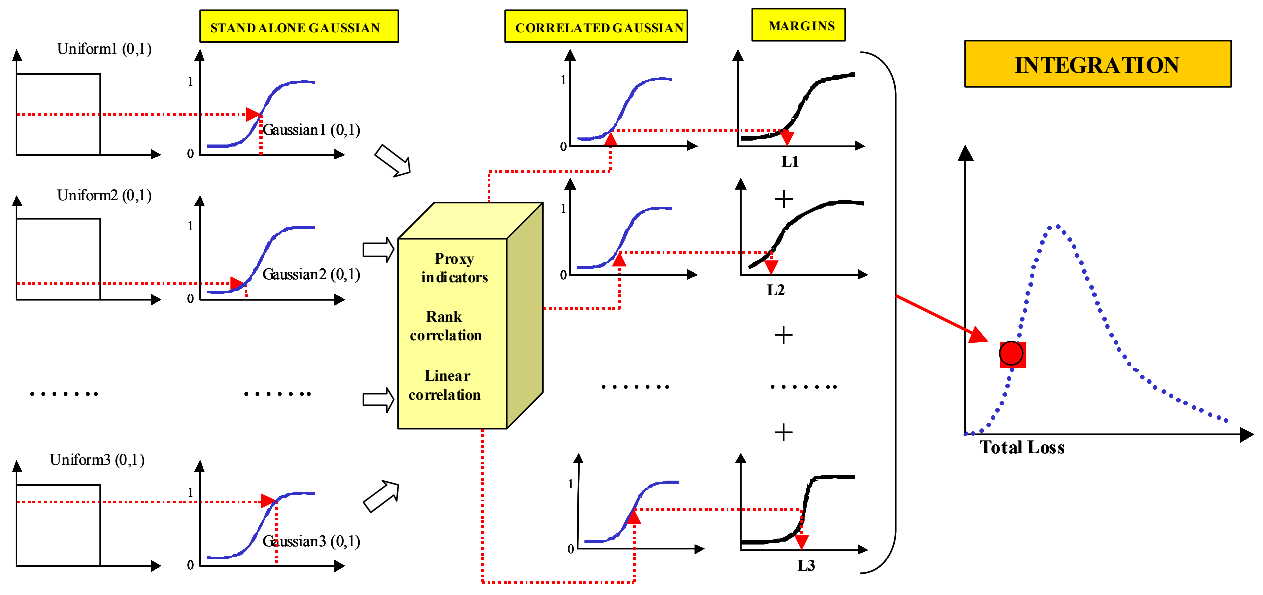
The process of obtaining the aggregate distributions through a Gaussian copula is detailed in Chart 2. We generate independent normal random numbers (X), which we correlate through the correlation matrix[3] obtaining X^*. We then calculate the normal cumulative probabilities $\phi(X^*)$ in order to recover the arguments of $C_\rho^{Ga}(\boldsymbol{u})$. Finally, the x_i (i.e. the loss of risk-i) are determined by inverting the marginal distributions F_i: $x_i = F_i^{-1}(u_i)$. Iterating this process and summing up each time the losses x_i, we trace the whole integrated distribution, as showed in the chart.

To simulate from a meta-t distribution, with ν degrees of freedom, the procedure is similar, apart from the fact that, after having obtained the correlated normal random numbers X^*, we still need to simulate an additional random variable y from the χ_υ^2 distribution, which is independent from

[3] Thanks to the eigenvalue algorithm suggested by Rousseeuw and Molenberghs (1993), as Cholewsky decomposition is not possible being ρ not positive definite.

vector X^*, and use the transformation $Z = (\upsilon/y)^{0.5}X^*$ to derive t-distributed random numbers. The arguments of $C_\rho^t(\mathbf{u})$ are thus obtained calculating the t-cumulative probabilities $t(Z^*)$.

CHART 2 Simulation mechanics in meta-Gaussian framework and aggregation of risks.



2.3 The correlation structure

The dependence measures used in the parametrization of copulas should not depend on the marginal distributions, but on the copula only.

Linear correlation can be safely used only in the case of elliptical models, like the Gaussian or the t copula. Where the marginal distribution functions are arbitrarily chosen (like in the meta-distributions), the non-linear dependence between the variables is not captured by linear correlations, which do not include the extreme cases.

In order to describe the dependence structure in the case of non-elliptical distributions, a practical solution is to use rank correlations, which are measures depending only on the copula of the multivariate distribution and not on the marginal distributions (unlike linear correlations which depends on both), and are then suited to calibrate the copula to empirical data.

In the present application the Kendall's τ coefficient was used, which is a measure of concordance for bivariate random vectors. Let's suppose that two independent vectors of random variables (X_1, X_2) and (X^*_1, X^*_2) are generated from the same distribution: ρ_τ can thus be defined as the probability of concordance minus the probability of discordance of the two pairs:

$$\rho_\tau(X_1, X_2) = P[(X_1 - X^*_1)(X_2 - X^*_2) \succ 0] - P[(X_1 - X^*_1)(X_2 - X^*_2) \prec 0] \qquad (9)$$

For pairs of sample series (e.g. series X_1 e X_2) not serially correlated, the standard estimator of ρ_τ is obtained through the method-of-moments:

$$\rho_\tau(X_1, X_2) = \binom{n}{2}^{-1} \sum_{t=1}^{n} \sum_{s=t+1}^{n} sign\left((X_{t,1} - X_{s,1})(X_{t,2} - X_{s,2})\right) \tag{10}$$

with "*sign*" being the value of the sign of the product (that is +1 or –1); ρ_τ is the average of the signs (+1 or –1) as determined by all the possible movements of the two observed variables.

The relationship between Kendall's Tau (ρ_τ) and Pearson's linear correlation ρ, thanks to elliptical copulas geometry, is the following one[4]:

$$\rho_\tau(X_1, X_2) = \frac{2}{\pi} arcsin(\rho) \tag{11}$$

Using this formula it becomes feasible, knowing the value of ρ_τ, to infer ρ without estimating the variances and covariance of the two series. If the ρ values obtained by inverting Equation (11) are used to calibrate the copula, the simulated numbers exactly replicate the same rank correlations which are used to generate them, which is not the case for linear correlations computed through variances and covariances. Furthermore, the Kendall's τ is a more efficient estimator than the linear ρ, especially if the underlying historical risk distributions are skewed and/or fat tailed[5].

This technique is useful to aggregate heterogeneous risks through copula functions, when the marginal functions are well understood thanks to the internal model but when information on correlations is lacking, due to the short historical period and the need to use proxy values for losses.

2.4 Reallocation of capital to the sources of risk

The methodology to calculate the capital contribution of each risk source (Window Conditional Expectation – WCE) is similar to the Expected Shortfall contribution, except that losses are averaged not on the whole interval of real numbers above a chosen percentile, but on a bounded interval.

The total WCE is thus:

$$WCE = E\left(L_T \mid VaR(L)_{\alpha-\Delta} \le L_T \le VaR(L)_{\alpha+\Delta}\right) \tag{12}$$

while WCE_i is the Expected Loss for risk i, conditional to the total loss values L_T in the interval between percentile α+Δ and percentile α-Δ.

$$WCE_i = E\left(L_i \mid VaR(L)_{\alpha-\Delta} \le L_T \le VaR(L)_{\alpha+\Delta}\right) \tag{13}$$

The additivity property, or the fact that the aggregate risk coincides with the sum of individual risks, holds:

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[4] McNeil, A. J., R. Frey, P. Embrechts [2005]

[5] See Lindskog [2000].

$$WCE = \sum_{i=1}^{n} WCE_i \tag{14}$$

where n is the number of the different risk types. This allows us to allocate back the simulated losses to the individual risks sources in a consistent way. The chosen window is symmetrical with respect to the percentile corresponding to capital; both on its left side and on its right side, the same number of basis points determines the limiting percentiles of the window.

Before selecting this methodology, other risk allocation measures were tested. The first example was the Conditional VaR suggested by D.Tasche[2000]:

$$\frac{\partial VaR(L_T)_\alpha}{\partial \omega_i} = E\left(L_i \mid L_T = VaR(L_T)_\alpha\right) \text{ [6]} \tag{15}$$

In our application L_T isn't differentiable and therefore not expressed analytically. Although we were able to reach a reasonable approximation using the right-hand part of Equation (15) with a large number of simulations, this technique was discarded due to instability of the results.

The second measure tested was the VaR-matched Expected Shortfall (explained by C. Bluhm, L. Overbeck, C. Wagner, 2003), a heuristic but coherent risk measure developed to preserve the VaR based overall economic capital. The aim of this method is to find a percentile *c* in order to guarantee the following condition:

$$ES(L_T)_c \approx VaR(L_T)_\alpha \tag{16}$$

In this case, the sum of the single ES contributions equates the economic capital based on VaR, taking into account the risks' tail shape and thus overweighting fat tailed risk-types such as operational risk.

After assessing the performance of each metric, WCE was determined to provide the most accurate and stable results, and hence selected as the measure to be used for the purpose of allocation.

3 RESULTS

The method described in the previous section is applied here to a stylized portfolio representative of a typical commercial bank with a small component of insurance activity, tailored to the Ecap composition of international benchmarks, in order to compute a total aggregated measure of diversified economic capital and to allocate back this diversified economic capital to each source. The time horizon of risks is set to one year and the economic capital requirement is assessed to protect losses at the 99.96% level, which is equivalent to an "AA-" rating.

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[6] Where ω_i is the exposure weight of risk i.

3.1 Marginal distributions

For the purpose of this exercise, nine risk types were considered, belonging to the macrocategories of credit (performing loans and defaulted loans), market (interest rate risk in banking book, trading book, equity, property), operating (business and operational) and insurance (not subdivided any further).

For many risk types it was possible to assume normality of the marginal distribution of losses, either because this assumption was consistent with the assumption made for computing stand-alone capital for the given risk type or because the assumption of normality was a reasonable approximation of the actual empirical distribution. Three risk types, however, were treated differently due to the fact that stand alone economic capital for these risk types was computed from simulation of losses. In particular, the loss distribution for performing loan credit risk losses were computed through Monte Carlo simulation of the loan portfolio asset values using Merton theory[7], the trading book loss distribution was obtained through historical simulation, and the operational risk loss distribution was based on an actuarial AMA model. Table 1 describes the main features of the different distributions involved in testing the methodology considered for this exercise. The last two columns show the stand-alone capital for each risk type, both in terms of absolute value and in terms of relative contribution. These results are consistent with benchmark studies that analyse typical weights of the different components of stand alone economic capital[8].

TABLE 1 Input marginal distributions figures[9].

Risk type	Distribution Type	$E(L_i)$	$\sigma(L_i)$	Skewness	Kurtosis	$VaR(L_i)_{99.96\%}$	%VaR
Performing loans	Empirical	0	443.0	3.6	26.2	4550	45.5%
Defaulted loans	Gaussian	0	119.3	0	0	400	4.0%
Banking	Gaussian	0	164.0	0	0	550	5.5%
Trading	Empirical	0	112.3	0.14	1.33	450	4.5%
Equity	Gaussian	0	104.4	0	0	350	3.5%
Property	Gaussian	0	89.5	0	0	300	3.0%
Business	Gaussian	0	268.4	0	0	900	9.0%
Operational	Empirical	76.6	255.5	248	74300	1500	15.0%
Insurance	Gaussian	0	298.3	0	0	1000	10.0%

In order to run the simulation which generates the aggregate distribution through the copula function, the marginal distributions that are empirically derived are made continuous by interpolating a cubic function among the percentiles.

[7] See Cuneo S. [2004].

[8] See for instance the Mercer Oliver and Wyman survey conducted in 2001.

[9] We assume that only for operational risk the mean contributes to economic capital.

3.2 Correlations

Correlations were computed based on proxy indicators selected to be representative of the different risk categories and derived from either internal data or market data when internal data were not available. The monthly series were observed over the period 2002-2005. Table 2 provides an overview of the proxy indicators used.

TABLE 2 Proxy indicators of risks for correlation estimation.

Risk type	**Indicators type**	**Source**
Performing loans	European credit spread changes, mapped on the bank portfolio.	Market
Defaulted loans	Market LGD implied in bond and bank loans price after default (Moody's data).	Market
Banking	Changes in price of a synthetic bond, built using asset and liabilities duration of the bank.	Market-Internal
Trading	Trading book's P&L.	Internal
Equity	Equity composition index.	Market-Internal
Property	Real estate price index.	Market
Business	Changes in bank's equity price, net of other risks effect	Market
Operational	Operational P&L.	Internal
Insurance	Insurance equity index.	Market

The choice to use external indices as proxies for the behaviour of particular risks, to overcome data insufficiency or inadequacy, is common among the banking institutions, as shown the Basel joint forum study on risk integration and aggregation (Basel Committee on Banking Supervision [2003]).

Table 3 provides the linear correlations matrix obtained using the various risk indices, while table 4 presents the rank inferred linear correlations.

TABLE 3 Linear correlations on approximated historical risk (2002-2005).

Risk type	Credit	Credit defaulted	Banking	Trading	Equity	Property	Business Risk	Op. Risk	Insurance
Credit	100%	**81%**	27%	9%	**46%**	-3%	**50%**	23%	**34%**
Credit defaulted	**81%**	100%	22%	7%	**37%**	-2%	**41%**	19%	**28%**
Banking	**27%**	22%	100%	7%	18%	9%	15%	11%	9%
Trading	9%	7%	7%	100%	**34%**	12%	**37%**	-15%	28%
Equity	**46%**	**37%**	18%	**34%**	100%	-7%	**87%**	-1%	**86%**
Property	-3%	-2%	9%	12%	-7%	100%	-7%	-18%	1%
Business Risk	**50%**	**41%**	15%	**37%**	**87%**	-7%	100%	7%	**82%**
Operational Risk	23%	19%	11%	-15%	-1%	-18%	7%	100%	-1%
Insurance	**34%**	**28%**	9%	28%	**86%**	1%	**82%**	-1%	100%

The bold coefficients are significatively different from zero (at the 95% confidence level).

Linear correlations derived from historical data and the ones derived from rank correlations are almost always of the same magnitude. In both cases, property risk and operational risk do not seem to be meaningfully correlated with the other risk types.

TABLE 4 Linear correlations implicit in rank correlations (Kendall's τ).

Risk type	Credit	Credit defaulted	Banking	Trading	Equity	Property	Business Risk	Op. Risk	Insurance
Credit	100%	81%	31%	8%	22%	0%	34%	-6%	26%
Credit defaulted	81%	100%	25%	6%	18%	0%	28%	-5%	21%
Banking	31%	25%	100%	16%	14%	12%	11%	6%	9%
Trading	8%	6%	16%	100%	42%	11%	48%	-4%	37%
Equity	22%	18%	14%	42%	100%	-4%	84%	-17%	85%
Property	0%	0%	12%	11%	-4%	100%	-7%	0%	6%
Business Risk	34%	28%	11%	48%	84%	-7%	100%	-11%	78%
Operational Risk	-6%	-5%	6%	-4%	-17%	0%	-11%	100%	-10%
Insurance	26%	21%	9%	37%	85%	6%	78%	-10%	100%

In order to be more conservative, the correlations used within the simulation are not the ones presented in Table 4, which are average measures, but instead the linear correlations obtained by stressing rank correlations estimates at the 99% confidence level. This approach was meant to mitigate the concern that the model may be inplausible in times of stress as parameter values (including correlations) might change under stressed conditions. As a result, and due to the short period of historical data, we decided to stress rank correlation using their standard error and normality property: $\rho(\mathbf{99}\%)_{\tau} = \rho_{\tau} + \mathbf{2.33} \cdot SE(\rho_{\tau})$. Table 5 presents the stressed values for the inferred linear correlations:

TABLE 5 Linear correlations implicit in rank correlations stressed at the 99% confidence level.

Risk type	Credit	Credit defaulted	Banking	Trading	Equity	Property	Business Risk	Op. Risk	Insurance
Credit	100%	100%	64%	44%	56%	37%	66%	31%	60%
Credit defaulted	100%	100%	64%	44%	56%	37%	66%	31%	60%
Banking	64%	64%	100%	51%	49%	48%	47%	43%	45%
Trading	44%	44%	51%	100%	72%	47%	77%	33%	69%
Equity	56%	56%	49%	72%	100%	34%	98%	21%	98%
Property	37%	37%	48%	47%	34%	100%	31%	37%	42%
Business Risk	66%	66%	47%	77%	98%	31%	100%	26%	97%
Operational Risk	31%	31%	43%	33%	21%	37%	26%	100%	28%
Insurance	60%	60%	45%	69%	98%	42%	97%	28%	100%

As a final test of reasonableness, correlation results were checked against the study conducted by Kuritzkes, Schuermann and Weiner (2002), which collected and analyzed various correlation sets from other documents in order to derive the diversification benefit under a plurality of hypotheses. The comparison between these (30%-80% for Credit-Market/ALM risks, 20%-44% for Credit-Operational/Other risks, 13%-40% for Market/ALM-Operational/Other risks), and the ones obtained here, shows a comfortable degree of similarity, thus indicating a good alignment with market benchmarks.

3.3 Aggregation of risks

Two copula dependence structures were tried for aggregating risks, namely the meta-Gauss and the meta-*t* copula with 3 degrees of freedom. Their properties were studied, in terms of stability, by repeating a 2.000.000 simulation exercise 50 times[10]. Table 6 compares the two resulting average outcomes in terms of diversified total economic capital to the economic capital which is obtained applying a variance / covariance approach (using the stressed correlation matrix):

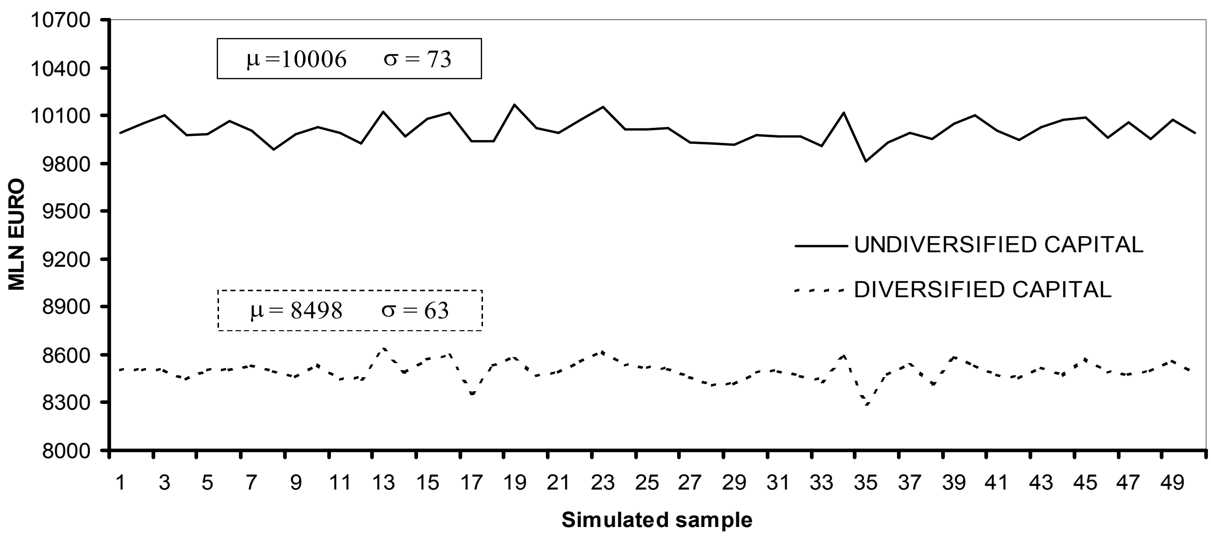
TABLE 6 Comparison among diversified economic capitals with 99.96 percentile.

	$VaR(L)_{99.96\%}$ stand alone		$VaR(L)_{99.96\%}$ diversified		RATIO	
APPROACH	MEAN	SD	MEAN	SD	MEAN	SD
Var/CoVar	10.000	-	8.009	-	80.1%	-
Gauss copula	10.011	54	7.613	49	76.0%	0.43%
t(3) copula	10.006	73	8.498	63	84.9%	0.40%

As previously discussed, the variance/covariance approach does not require simulation, as it uses the closed formula (1) *à la* Markowitz. Both copula dependence structures allow us to reach a good degree of convergence, in terms of standard deviation computed on the 50 simulated samples. The stand alone VaR does not exactly coincide with the input data, as it is itself an outcome of the simulation (or the sum of the 99.96th percentiles of the simulated marginal distributions of risk).

Chart 3 compares, in the case of the *t* dependence structure, the diversified and undiversified capitals in each of the 50 simulated samples:

CHART 3 Comparison between diversified and undiversified Capital under meta-*t*(3) copula; the graph shows the mean and standard deviation calculated on 50 random sample.



The meta-*t* copula approach is the most conservative, with a diversification effect of 15%, lower than in the var/covar approach (20%) and in the Gauss copula application (24%). It is probably also

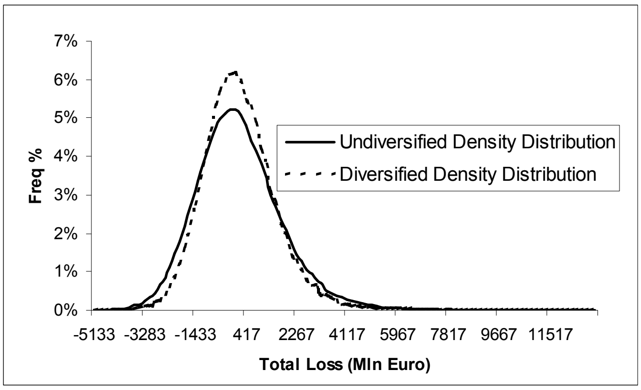
[10] Programming with SAS/IML language, with a computational cost of one hour for meta-gaussian copula and about one hour twenty for meta-*t*.

the most realistic one, as it both takes into account the true marginal distributions of risks (as also the Gauss copula does) and assumes dependence among extreme events (which is a feature of t distributions). Joint extreme events are in fact quite typical in a portfolio where fat tails of credit and operational risks could exercise a relevant contagion effect on the other risk types, thus justifying the choice of the t dependence structure.

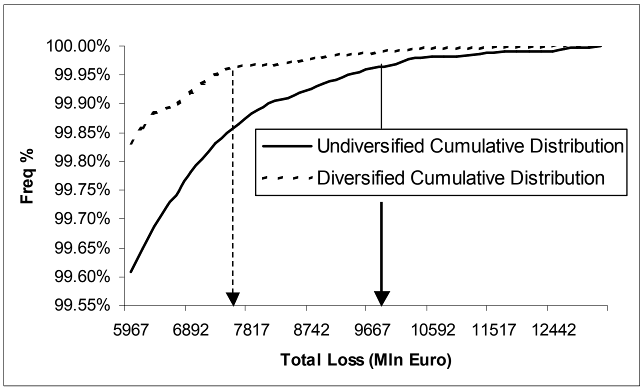
The diversification effect can be most easily appreciated on single samples: chart 4 shows the difference between the joint distribution and the sum of the simulated distribution (this last one being equivalent to a 100% correlation hypothesis) in the case of a specific sample, randomly chosen, out of the 50 that were simulated, for both the Gauss and the *t* dependence structures.

CHART 4 Empirical loss distribution and analysis of percentile 99.96 under meta-gauss and meta-*t*(3) copula in one random sample.

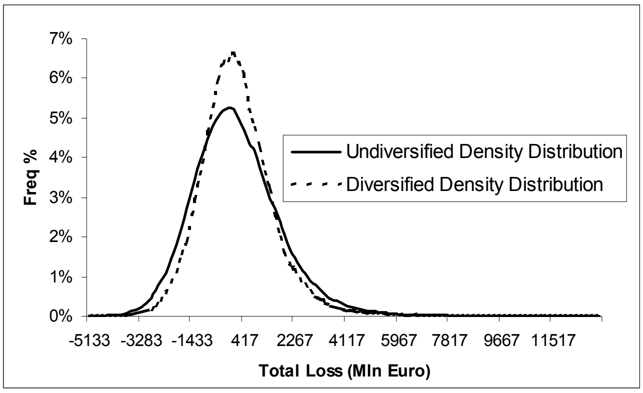
Density distribution through meta-gauss copula



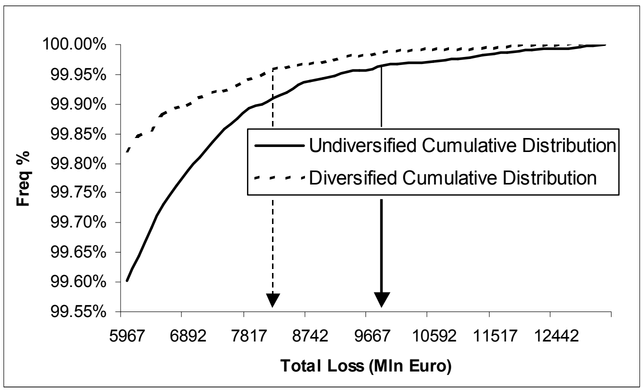
Cumulative distribution through meta-gauss copula



Density distribution through meta-*t*(3) copula



Cumulative distribution through meta-*t*(3) copula



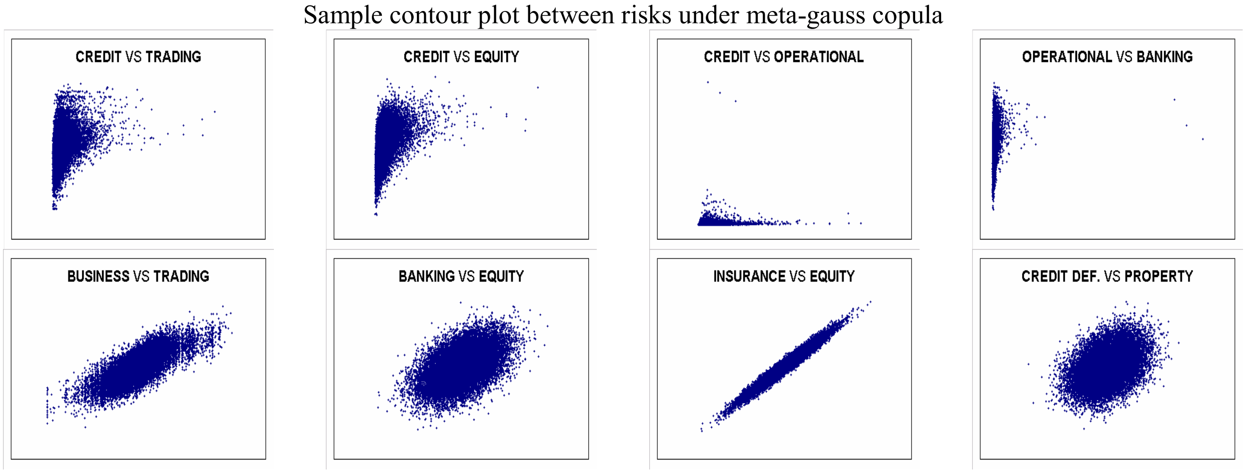
Using the same sample, we could verify the invariance of the rank correlations. However, the values of Pearson linear correlation calculated on the simulated risks (Table 7), if compared with the input ones presented in table 5, deviate significantly in the case of credit and operational risks as the shape of the joint distributions of those risks with the other risks is in fact clearly not elliptical (see Chart 5 for examples about the Gauss and the *t* cases).

TABLE 7 Linear correlations calculated on a random sample of 2'000'000 simulations using *t*(3) copula.

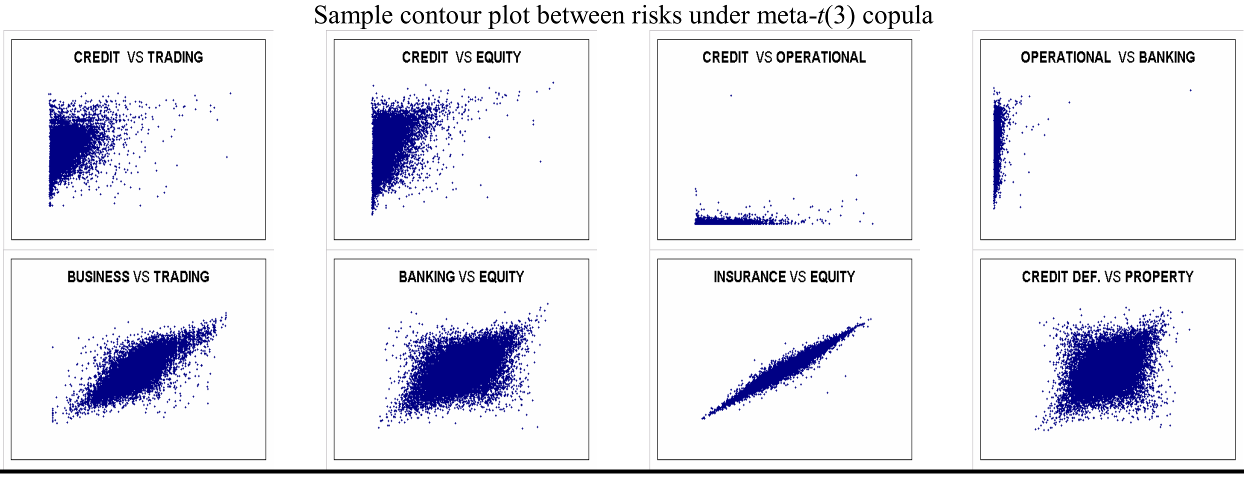
Risk type	Credit	Credit defaulted	Banking	Trading	Equity	Property	Business Risk	Op. Risk	Insurance
Credit	100%	85%	54%	38%	47%	31%	56%	13%	51%
Credit defaulted	85%	100%	63%	43%	55%	36%	65%	6%	59%
Banking	54%	63%	100%	50%	48%	47%	46%	8%	44%
Trading	38%	43%	50%	100%	71%	46%	76%	7%	68%
Equity	47%	55%	48%	71%	100%	33%	97%	4%	98%
Property	31%	36%	47%	46%	33%	100%	30%	7%	41%
Business Risk	56%	65%	46%	76%	97%	30%	100%	5%	97%
Operational Risk	13%	6%	8%	7%	4%	7%	5%	100%	6%
Insurance	51%	59%	44%	68%	98%	41%	97%	6%	100%

CHART 5 Contour plot of simulated risk on a random sample of 20'000 simulations using gauss and *t*(3) copula.

Sample contour plot between risks under meta-gauss copula



Sample contour plot between risks under meta-*t*(3) copula



Having reviewed the relevant literature, our results appear to be consistent with the empirical results of other related studies. For example, in the study of Berg-Yen and Medova (2004), data collected from a sample of bank's economic capital numbers indicate a diversification effect ranging from a minimum of 7.2% to a maximum of 33.49% (the range is obviously influenced by different risk profile and different measurement and aggregation methodologies). Furthermore, Dimakos and Aas (2003) find, in their work for a Norwegian bank, a diversification benefit of 20%.

3.4 Capital allocation to risk types

The capital contribution calculated on the basis of the individual risks contribution to the Window Conditional Expectation, as an average on the 50 simulated 2.000.000 scenarios samples, is shown in Table 8 ("sample div. ECAP" column).

TABLE 8 Inter-risk allocation with copula *t*(3) model using $WCE_{\Delta=3bp}$.

Risk type	Und. ECAP	Initial comp.	Sample und. ECAP	Sample div. ECAP	Final comp.	Div. ECAP / und. ECAP	SD	SE
Credit	4550	45.5%	4551	4351	51.2%	95.6%	0.7%	0.1%
Credit defaulted	400	4.0%	400	378	4.4%	94.4%	0.6%	0.1%
Banking	550	5.5%	550	445	5.2%	80.9%	0.9%	0.1%
Trading	450	4.5%	450	332	3.9%	73.8%	1.3%	0.2%
Equity	350	3.5%	350	289	3.4%	82.7%	0.9%	0.1%
Property	300	3.0%	300	177	2.1%	59.1%	1.6%	0.2%
Business Risk	900	9.0%	900	790	9.3%	87.8%	0.8%	0.1%
Op.Risk	1500	15.0%	1504	881	10.4%	58.6%	2.2%	0.3%
Insurance	1000	10.0%	1000	855	10.1%	85.5%	0.9%	0.1%
Total	10000	100.0%	10006	8498	100.0%	84.9%	0.4%	0.1%

ECAP is based on 99.96% VaR.

The chosen Δ was 3 bp, which guarantees a sufficient stability of results. The amount of the diversification effect ranges from 4.4% in the case of credit risk to 41.4% for operational risk ("diversified capital / stand alone capital" column, calculated as a ratio to the sample undiversified capital). Diversification effect results are mainly driven by credit risk correlations, as credit risk represents the most relevant source of risk in the portfolio. This fact explains, for instance, the strong diversification effect observed for operational risk, which is the least correlated risk type (with credit and also with the other risk types).

The standard deviation and the standard error ("SD" and "SE" columns) of the operational risk contribution are the highest among risks, due to the fatter tails of the operational losses distribution.

Tables 9 and 10 compare the meta-*t* copula aggregation and WCE (3bp window) re-allocation results with alternative diversified capital contributions. Some of the major points that emerge from the comparison include:

- If a Gauss copula is used instead of the t-copula, the Window Conditional Expectation leads to a significantly different economic capital risk composition. The diversification effect becomes in fact larger for less volatile marginal distributions (for instance trading, equity or property) and for operational risk, whose low correlation with the other risks is no longer counterbalanced by the tail dependency effect, which is asymptotically ignored by the Gauss copula.

TABLE 9 Comparison between Var/Covar approach, *t*(3) and Gauss copula allocation using $WCE_{\Delta=3bp}$. Average results deriving from 50 random samples.

Risk type	Initial comp.	Var/Covar Final Comp.	Var/Covar Div. ECAP / und. ECAP	$WCE_{\Delta=3bp}$ *t*(3) copula Final Comp.	$WCE_{\Delta=3bp}$ *t*(3) copula Div. ECAP / und. ECAP	$WCE_{\Delta=3bp}$ *t*(3) copula SE	$WCE_{\Delta=3bp}$ Gauss copula Final comp.	$WCE_{\Delta=3bp}$ Gauss copula Div. ECAP / und. ECAP	$WCE_{\Delta=3bp}$ Gauss copula SE
Credit	45.5%	53.0%	93.2%	51.2%	95.6%	0.1%	57.9%	96.8%	0.1%
Credit defaulted	4.0%	4.7%	93.2%	4.4%	94.4%	0.1%	5.0%	95.0%	0.1%
Banking	5.5%	5.0%	72.2%	5.2%	80.9%	0.1%	4.9%	67.7%	0.1%
Trading	4.5%	3.6%	64.7%	3.9%	73.8%	0.2%	3.0%	50.6%	0.1%
Equity	3.5%	3.3%	74.9%	3.4%	82.7%	0.1%	3.0%	65.4%	0.1%
Property	3.0%	1.9%	49.7%	2.1%	59.1%	0.2%	1.7%	43.3%	0.1%
Business Risk	9.0%	9.2%	82.0%	9.3%	87.8%	0.1%	8.7%	73.6%	0.1%
Op.Risk	15.0%	9.6%	51.4%	10.4%	58.6%	0.3%	6.7%	33.8%	0.4%
Insurance	10.0%	9.8%	78.5%	10.1%	85.5%	0.1%	9.1%	69.2%	0.1%
Total	100.0%	100.0%	80.1%	100.0%	84.9%	0.1%	100.0%	76.0%	0.1%

- The conditional VaR re-allocation method would be the theoretically most consistent one, but its results, even if reasonable from the point of view of the diversification effect, are largely unstable. This is particularly evident for the fat-tailed operational risk, whose standard error is nearly 12%, meaning that the average contribution on other sets of 50 independent samples, each one simulating 2.000.000 vectors of risks, could be smaller or bigger than the first one by more than 24% (2 standard errors) in 5% of cases.
- For the VAR matched Expected Shortfall, the two columns showing the diversification effect are calculated respectively on the stand alone VaR and on the stand alone Expected Shortfall resulting from the simulation. They strongly differ as far as credit risk and operational risk are concerned. Effects calculated on the stand alone Expected Shortfall would of course be theoretically more consistent, and its results in terms of diversification appear reasonable. It has nevertheless the effect of changing the stand alone capital for each risk type, which is not acceptable being an input of the whole framework. On the other hand, the diversification effect computed on the stand alone VaR is relatively small for operational risk, and conversely relatively high for credit risk.

TABLE 10 Comparison between Conditional $VaR(L)_{99.96\%}$ and $VaR(L)_{99.96\%}$ matched Expected Shortfall under *t*(3) copula. Average results deriving from 50 random samples.

Risk type	Initial comp.	Conditional VaR *t*(3) copula Final Comp.	Conditional VaR *t*(3) copula Div. ECAP / und. ECAP	Conditional VaR *t*(3) copula SE	VaR matched Expected Shortfall – *t*(3) copula Final comp.	VaR matched Expected Shortfall – *t*(3) copula Div. ECAP / und. ECAP	VaR matched Expected Shortfall – *t*(3) copula Div. ECAP / und. ES	VaR matched Expected Shortfall – *t*(3) copula SE
Credit	45.5%	50.9%	95.0%	2.8%	45.6%	85.1%	93.4%	0.0%
Credit defaulted	4.0%	4.6%	97.6%	1.0%	4.3%	92.1%	95.7%	0.0%
Banking	5.5%	5.3%	82.1%	4.8%	5.1%	78.4%	81.5%	0.1%
Trading	4.5%	3.7%	70.7%	6.6%	3.8%	72.0%	74.5%	0.1%
Equity	3.5%	3.1%	74.8%	5.2%	3.3%	80.5%	83.7%	0.1%
Property	3.0%	2.0%	55.3%	7.8%	2.0%	57.3%	59.6%	0.1%
Business Risk	9.0%	9.0%	84.8%	3.9%	9.1%	85.6%	89.0%	0.1%
Op.Risk	15.0%	12.1%	68.4%	11.8%	17.0%	96.1%	70.3%	0.2%
Insurance	10.0%	9.3%	79.4%	4.9%	9.8%	83.3%	86.6%	0.1%
Total	100.0%	100.0%	84.9%	0.1%	100.0%	84.9%	84.9%	0.0%

The results obtained here support the choice to use, for practical applications, the Window Conditional Expectation, which appears to be a fairly good compromise between theoretical soundness and plausibility of the calculated diversification.

3 CONCLUDING REMARKS

The methodology described in the present paper belongs to the top-level aggregation approaches, where marginal models for the loss distribution of each risk type are independently developed and then merged into a joint distribution. In this sense it belongs to the same category of the variance/covariance approach, as the difference between the two approaches only concerns the way the marginal distributions are merged (through a copula function in the present application or through a correlation structure in the alternative treatment of risks).

Shifting to a base-level aggregation approach would involve identifying the economic risk factors driving the different risk types and developing a simultaneous model for these factors. In this case, the primary dependency structure would be either a correlation matrix or a copula, while the marginal loss distributions would be indirectly correlated through the relationships between model factors.

Even if it is certainly worthwhile to deeper investigate the base-level approach, at the state of the art this couldn't substitute the top-level one, as knowledge about each individual risk is both useful for internal capital management and consistent with the Basel II framework, whose requirements are risk specific.

This paper's results show that, even with the very conservative hypotheses which were applied here (the meta-t copula, the stressed correlations and the fat-tailed distributions for credit and operational risk) we still attain a quite substantial diversification benefit, in terms of difference between undiversified and diversified economic capital.

The methodological approach we suggest for aggregating different sources of risk in an overall measure of diversified economic capital, which in turn can be allocated back to the single risks, focuses on three main issues: correlations derived from rank correlations instead than estimated through variances and covariances of the historical series; aggregate capital through a copula dependence structure, preferably one taking into account the tail dependence of the margins, as the t-copula; re-allocating risk through the Window Conditional Expectation, or the average of risk contribution in a closed interval around the percentile corresponding to capital.

Even if the presently applied approach is theoretically sounder than the standard framework *à la* Markowitz, this one, despite its over-simplistic assumptions which lead to a flawed representation of risks, gives numerical results which do not dramatically differ from the correct ones, and could thus represent a good approximation of the "true" risk interplay.

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