An Evaluation of the Base Correlation Framework for Synthetic CDOs

Søren Willemann*

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Abstract

In April 2004, JP Morgan introduced the notion of base correlations, a novel approach to quoting correlations for synthetic CDO tranches, which facilitates a simple relative valuation of off-market tranches. Using a simple intensity based credit risk model we generate ‘true’ tranche spreads and examine the behavior of the base correlations and the merits of the relative valuation approach. We reach four conclusions: First, even if the true default correlation increases, base correlations for some tranches may actually decrease. Second, we uncover a structural uniqueness problem with the definition of base correlations. Third, in the relative valuation framework expected losses can go negative for steep correlation skews. Fourth, the relative spread errors are small in some segments but can be very large, +/- 10% for mezzanine tranches, the errors changing sign from tranche to tranche.

Keywords: Credit risk, Default correlation, Collateralized Debt Obligation, Correlation skew

JEL codes: C52, E47, G12, G13

*Department of Accounting, Finance and Logistics, Aarhus School of Business, Denmark, swi@asb.dk. Mailing address: Aarhus School of Business, Department of Accounting, Finance and Logistics, Fuglesangs Alle 4, 8210 Aarhus V, Denmark. Phone: +45 2240 6209. I thank Leif B.G. Andersen, Jens Christensen, Mads Fredsgaard, David Lando, Allan Mortensen and Mikkel Svenstrup for helpful comments and discussions. This research was supported by ScanRate Financial Systems.
1 Introduction

In recent years, the market for Collateralized Debt Obligations (CDO) has boomed. A CDO is a security, which is backed by a pool of corporate bonds or loans and promises a cash flow where the risk of losses is divided into tranches of different seniorities. The prices of the respective tranches depend critically on the perceived likelihood of joint defaults of the underlying pool, or loosely speaking the default correlation.

Market participants have therefore been interested in a measure of correlation, which can be used as a quotation device to facilitate a comparison of prices across tranches. The line of thought is similar to Black-Scholes implied volatilities of equity markets. Even if the Black-Scholes model is simplistic, the implied volatilities facilitate a comparison of options across maturities and strikes and they have therefore become tremendously popular.

A direct extension from Black-Scholes implied volatilities to the CDO market is termed compound correlations. By inverting a standard pricing model for CDOs, it is possible to find the correlation matching the quoted spread of a tranche. A drawback with compound correlations is that they, for reasons we discuss later, are not generally unique.

Because of this, there has been increased attention on another type of correlation developed by JP Morgan, termed base correlations. The key benefit of base correlations is that the behavior is more like implied volatilities from equity options: base correlations are monotonic in correlation and thus unique. Further, in a spirit similar to implied volatilities, base correlations allow for a relative valuation of off-market tranches by interpolating in the base correlations.

This paper investigates the behavior of base correlations and we find that there are problems with their use, both as a quotation device and as a relative valuation tool. Using a theoretical model in which we can control the default correlation, we generate spreads of CDO tranches and examine how base correlations react in response to changes in the default correlation. We can summarize our findings in four points. First, even if tranche spreads from a theoretical model change to reflect an increased default correlation, the base correlations for some tranches can decrease. This means that if spreads in the market change as to reflect an increased default correlation, the base correlations may wrongfully indicate that correlations have decreased. Second, the relative valuation framework can yield negative losses for mezzanine tranches. Third, relative spread errors can be very large, +/- 10% for mezzanine tranches, the errors changing sign from tranche to tranche. Fourth, base correlations are only unique given the set of attachment points. This means that across the North American and European markets, base correlations will be different simply because the structure of traded tranches is not the same.

Before introducing the setup in which to analyze base correlations, let us provide some background knowledge of CDOs. It is not the purpose of this paper to provide a survey of the CDO market. Duffie & Garleanu (2001), Bluhm (2003) and Duffie
(2004) provide descriptions of the CDS and CDO markets, rationales for trading CDOs, examples of tranched products and economic intuition on various types of CDOs.

The traditional CDOs are termed cash CDOs, consisting of a pool of corporate bonds or loans, against which notes are then issued. The payment structure is tranched such that it is possible to buy exposure to a certain portion of the loss distribution of the pool of corporate bonds. For this exposure, the investor receives compensation in terms of claims to a fraction of the coupon payments paid to the CDO. The fraction of coupons is set to reflect the default risk in the tranche.

To understand the most popular kind of CDO, we need to introduce a more fundamental credit derivative. A Credit Default Swap (CDS) is a contract, which references a particular firm. In case of default of the firm, the seller of the swap (the protection seller) pays compensation. In return for this, the buyer of the swap (the protection buyer) pays a periodic spread until default or maturity of the swap, fixed at the initiation of the contract, termed the CDS spread.

Using CDSs, a synthetic CDO can be created. Instead of having corporate bonds in the pool, the credit risk is created synthetically by having default-free treasury bonds and CDSs in the pool. Then, instead of receiving fractions of coupon payments in tranches, the CDS spreads are received and instead of paying default losses on the corporate bonds, the compensation required by the CDSs is paid. The next step, moving towards the most recent types of CDOs, is to remove the treasury bonds from the CDO such that the underlying pool consists of CDSs only. This kind of CDO is unfunded; essentially, it means that there is no principal involved in the CDO and there is no up front payment. The base correlation framework we analyze here is applicable to any other type of CDO but the focus is on synthetic unfunded CDOs. We expect the findings to carry over to other CDO structures.

The structure of the paper is as follows. In section 2, we outline the standard model proposed by JP Morgan along with the notion of compound and base correlations. In section 3, we introduce the model generating our 'true' spreads. In section 4, we analyze the base correlation setup and the relative valuation tool and finally section 5 concludes.

2 A standard model

In this section, we outline the standard model we use for quoting correlations. To appreciate the problems with the traditional compound correlation approach outlined in the introduction, we start by discussing exactly how default correlation affects values of different tranches. We then proceed by introducing the model framework of JPM\textsuperscript{1}. We finalize this section by formally introducing compound correlations and

\textsuperscript{1}This model is not specific to JP Morgan, but merely suggested as a common framework for quoting correlations. Further, the model introduced here is not the model JP Morgan uses, or
2.1 Effect of correlations on tranches

In this section, we provide intuition on how default correlation affects the value of different tranches. We define a tranche by the upper and lower attachment point at which the tranche experiences losses, termed the upper and lower attachment points.

Let us focus on the tranche taking the first losses, the so-called equity tranche. The ’payoff’, in terms of loss on the CDS portfolio, has option-like features in the sense that the loss payment is of the form \( \min(L_{\text{portfolio}}, K_U) \). Here, \( L_{\text{portfolio}} \) is the loss in the portfolio and \( K_U \) is the upper attachment point, 3% for example. Throughout, \( L_{\text{portfolio}} \) is a positive random variable.

The equity tranche absorbs any losses below \( K_U \) and the more senior tranches absorb losses above that. Increased default correlation among the firms referenced by the CDSs, keeping the marginal default probabilities fixed, means that it becomes more likely to observe many or few defaults. Because of the upper limit on losses, the equity tranche is not affected (much) by occurrences with many defaults. On the other hand, there is upside in occurrences with few defaults, as the payments of the tranche holder would then decrease. This reduces the expected loss in the tranche and in turn, the fair spread. Thus, when correlation goes up, the expected loss decreases in an equity tranche.

Because of this, the fair spread is monotonic in the default correlation between the CDSs in the portfolio. This has an analogy in the equity derivatives literature where the value of a call option is increasing in the volatility of the underlying, due to the limited downside and unlimited upside of large swings in the value of the underlying.

Focusing instead on the senior tranche, we have the reverse relationship. Only losses above for example 22% of the pool affects this tranche. Thus, many defaults have to occur before it is affected. The probability of this event increases with increased correlation so the expected loss and fair spread of the senior tranche increase monotonically with correlation.

For mezzanine tranches, let \( K_L \) be the lower attachment point, and \( K_U \) the upper attachment point. The loss in the tranche is

\[
L_{\text{tranche}} = \min(L_{\text{portfolio}}, K_U) - \min(L_{\text{portfolio}}, K_L)
\]

such that, if \( L_{\text{portfolio}} < K_L \), the loss is zero, if \( K_L < L_{\text{portfolio}} < K_U \), the loss is \( L_{\text{portfolio}} - K_L \) and if \( L_{\text{portfolio}} > K_U \), the loss is \( K_U - K_L \). For both components in the expression above, the expected value is decreasing in the correlation in the loss portfolio. Since the components enter the expression with opposite signs, we cannot generally be sure that the expected loss in the tranche is monotonic in the correlation. This in turn means that we cannot expect fair spreads of mezzanine tranches to be monotonic in correlation.

advocates, for risk management of CDO tranches.
2.2 Model framework of JP Morgan

In this section, we outline the modeling framework proposed by JP Morgan. A thorough documentation is available in the four research papers, McGinty & Ahluwalia (2004a, b,c,d). The standard model is a so-called Homogeneous Large Pool Gaussian Copula Model (HLPGC hereafter) and descriptions are available in McGinty & Ahluwalia (2004c) and Schönbucher (2003) but we give a brief overview of the model here.

The model assumes that the reference portfolio of the CDO consists of an infinite number of firms, each with the same characteristics. Based on the assumption of homogeneity we can reduce the problem to a single representative firm and assume that default occurs if

\[ X_n < C \]

where \( X_n \) is a normally distributed random variable and \( C \) is a constant boundary. \( X_n \) can be interpreted as the firm value and \( C \) as the level of liabilities and default occurs when the firm value is below the liabilities.

Assuming a fixed recovery rate, we can choose \( C \) such that the default probability implied by the equation coincides with the default probability implied by a quoted CDS spread. In practice, one can use the mid quote of a DJ TRAC-X CDS and assume a fixed recovery rate of 40%.

We achieve correlation in the model by assuming that

\[ X_n = \sqrt{a}Z + \sqrt{1-a}\varepsilon_n \]

where \( a \) is a real number, \( Z \) is a common component, normally distributed, and \( \varepsilon_n \) is the idiosyncratic component, also normally distributed. In this setup, \( a \) is the default correlation.

Using the assumed values of \( C \) and \( a \), we can calculate the expected losses in a given tranche. We then annualize the expected loss over the maturity of the CDO and from this, we can calculate the fair spread of a particular tranche.

Unfortunately, the convenient and intuitive model described above does not conform to market data. Fixing a specific correlation parameter and calculating tranche spreads, the errors are very large. Instead, practitioners use the model as a quotation device, deriving implied correlations as described in the next section.

2.3 Implied correlations

Since the default correlation is such an important driver for the relative value of tranches, it is becoming market practice to use a standard model as above as a quotation mechanism. In doing so, spreads of different tranches and products can be compared more easily. The inspiration is taken from the equity options market where the Black-Scholes implied volatilities provide intuition about the relative value, even if there is a general agreement that the model is incorrect.
In the so-called *compound correlation* approach, a tranche is fixed. The pricing formula is then numerically inverted so the default correlation parameter $a$ in equation (2) can be chosen to yield the quoted spread. It usually turns out that when calculating compound correlations, the correlation required to price each individual tranche correctly is not constant. Instead, it seems that more senior tranches trade as if the correlation in the standard model is higher. This difference in implied correlations across tranches gives rise to the term *correlation skew*. In some ways, it can be thought of as the equivalent of an implied volatility smile of equity options.

As discussed above, the equity tranche spread is monotonic in the correlation and so is the senior tranche. Hence, for these tranches, the compound correlation works well. However, the lack of monotonicity for mezzanine tranches creates problems: provided a solution exists at all, there can be two solutions. For an excellent example of this, see McGinty & Ahluwalia (2004a).

To alleviate this problem, McGinty & Ahluwalia (2004b) of JPM have developed a new type of correlations called *base correlations*. This type takes its foundation in the monotonicity of equity tranches and extends this by including additional, fictive, equity tranches, which can be used to construct the traded mezzanine tranches.

Formally, consider a tranche with lower and upper attachment points $K_L$ and $K_U$ and assume that an equity tranche with upper attachment points $K_L$ is traded. Then

$$E[L(K_L, K_U)] = E[L(0, K_U)] - E[L(0, K_L)]$$  \hspace{1cm} (3)

illustrates how the expected loss of a mezzanine tranche, on which the JPM framework bases its fair spread calculation, can be decomposed into the expected losses of two equity tranches. The line of thought is very similar to the example of equation (1).

For the $(0, K_L)$ tranche, we can, given the fair spread, back out the unique correlation, which generates this spread. We denote this correlation the *base correlation for attachment point* $K_L$. Note that this is the same as the compound correlation. Given the $K_L$ base correlation we fix the expected loss in the equity tranche $E[L(0, K_L)]$ of equation (3) and given the fair spread of the $(K_L, K_U)$ tranche we iterate over the correlation parameter $a$ in equation (2) which generates an expected loss of a fictive $(0, K_U)$ tranche such that the expected loss of the $(K_L, K_U)$ tranche via equation (3) implies the given spread. This is denoted the base correlation of attachment point $K_U$ and it is thus the unique correlation of a $(0, K_U)$ tranche which given the base correlation of attachment point $K_L$ is consistent with the quoted spreads. We can proceed in this fashion and extract base correlations for all attachment points of traded tranches.

Briefly speaking, the base correlation approach seeks to exploit the monotonicity of equity tranches to construct fictive equity tranches, consistent with the observed tranche spreads. This is done via a bootstrapping mechanism through equation (3) and results in a set of unique correlations.

We emphasize that the notion of base correlation is not unique to the HLPGC model but is usable in conjunction with any model for CDO spreads, even the
intensity-based model introduced in the next section.

Embedded in the base correlation framework is a very convenient method to value off-market tranches based on the traded tranches. This is simply done by interpolating to find base correlations for the upper and lower attachment points of the off-market tranche and then calculate expected loss of the two equity tranches and thereby the expected loss in the tranche, via equation (3), and the fair spread.

Having described the standard model and how we can use it to extract base correlations, we can proceed with the analysis. In the next section, we introduce the model for default events, which we take as the real world. At the same time, we describe how we can use the model to calculate fair CDS spreads and spreads for CDO tranches. Within the model, we can control the default correlation, which will enable us to understand the mechanisms of the base correlation framework.

3 Generation of "true" spreads

In this section, we introduce the model for default events, which we take as the real world. Using this model, we show how to calculate fair spreads of CDSs and CDO tranches.

As in Duffie & Garleanu (2001), we use an intensity-based setup to model the losses in the portfolio, but we use a simpler specification of the intensity process. Let us start with some preliminaries, based mainly on Lando (1998). Despite being important for the theoretical results used here, we do not delve into technical details; these are available in for example Lando (1998).

3.1 Preliminaries

We assume the existence of a risk neutral pricing measure $Q$, under which all price processes discounted with the interest rate process $r$ are martingales. All expectations are with respect to this measure.

Let us initially consider a single firm where the default time is given by the stopping time $\tau$. We define $\tau$ to be the first jump time of a doubly stochastic Poisson process with intensity $\lambda$, which means that the intensity of the process in itself is a non-negative stochastic process. Throughout we invoke the recovery of treasury assumption stating that in the event of default, the recovery rate is a fraction $R$ of the face value of the claim, received at maturity of the contract, and we assume that $R$ is deterministic.

When valuing CDOs we resort to Monte Carlo simulation and within the simulation, we need to calculate the default time. To do this we postulate, see Lando (1998), that for a stochastic variable $U \sim U(0, 1)$ the stopping time defined by

$$\tilde{\tau} = \inf \left\{ t \left| - \int_0^t \lambda_u du \leq \ln U \right. \right\}$$

(4)
is distributed in the same way as \( \tau \). To see this, let \( F_U \) be the \( U(0,1) \) distribution function and calculate

\[
\mathbb{Q}(\tilde{\tau} > t) = \mathbb{Q}(\ln U < - \int_0^t \lambda_u du) \\
= \mathbb{Q}(U < \exp(- \int_0^t \lambda_u du)) \\
= F_U\left( \exp\left(- \int_0^t \lambda_u du\right) \right) \\
= \exp\left(- \int_0^t \lambda_u du\right) \\
= \mathbb{Q}(\tau > t)
\]

due to properties of Cox processes.

Therefore, to simulate the default time we can simply draw a uniformly distributed number \( U \) and then simulate the intensity process. We then keep track of when, or if, the condition in (4) is fulfilled. The first time this happens is the default time of the firm.

### 3.2 Valuation of a CDS

To value CDOs in the JPM setup one of the inputs is the average CDS spread of the underlying portfolio, which we also need to calculate. We define a CDS by the following: the reference firm with default intensity \( \lambda \), the maturity of the contract \( T \), the payments per year \( N \), \( \delta = 1/N \), and the annualized spread \( s \), which we want to calculate. We assume \( T \) is an integer, easily generalizable to any real number, and define \( n = T \cdot N \) and \( t_i = i\delta \). The protection buyer pays the spread \( s \) to the protection seller until the time of default or maturity, whichever comes first. In the case of default, the protection buyer delivers the defaulted asset and the protection seller pays the face value. In our setup, this means that the protection seller pays \( 1 - R \) per unit of face value.

Like traditional swaps, a CDS consists of two legs: the payment leg PL and the default leg DL. Assuming that the spread \( s \) is given, the values of the two legs are

\[
DL = (1 - R) E\left[ \exp\left(- \int_0^\tau \r_t dt\right) 1_{\{\tau \leq T\}} \right] \\
= (1 - R) E\left[ \int_0^T \lambda_t e^{-\int_0^\tau (\r_u + \lambda_u) du} dt \right]
\]
and

\[ PL = E \left[ \sum_{i=1}^{n} s \delta 1_{\{r > t_i\}} \exp \left( - \int_{0}^{t_i} r_u du \right) \right] \]

\[ = E \left[ \sum_{i=1}^{n} s \delta \exp \left( - \int_{0}^{t_i} (r_u + \lambda_u) du \right) \right] \]

where the change from the default indicator under both expectation operators are due to results of Lando (1998).

The fair spread is determined such that the values of the two legs are equal. This means that \( s \) is equal to

\[ s = (1 - R) \frac{E \left[ \int_{0}^{T} \lambda_t e^{-\int_{0}^{t} (r_u + \lambda_u) du} dt \right]}{E \left[ \sum_{i=1}^{n} \delta \exp \left( - \int_{0}^{t_i} (r_u + \lambda_u) du \right) \right]} \]

Therefore, to find the fair spread for a CDS we need to calculate the two expectations in the above expression. For very simple specifications of the intensity process, constant or affine for example, this can be done in closed form while for processes that are more complicated we need to resort to the solution of partial differential equations or Monte Carlo simulation. As the focus here is on CDOs, we do not delve into how we can evaluate the two expectations as the approach is very similar to the one used for CDOs, which we describe in detail next.

### 3.3 Valuation of a CDO

In this section, we introduce the valuation setup of a CDO. We focus on the valuation of a synthetic unfunded CDO, which means that the underlying pool of assets consists solely of CDSs. We can easily extend the valuation setup to accommodate funded CDOs.

We start by introducing the notation and subsequently outline the steps in a Monte Carlo simulation used to value the CDO. Table 1 introduces the notation, divided into three main categories: contractual setup of the CDO, default and recovery and finally distribution of losses among the tranches.

We let \( D_t \) be the discount factor for maturity \( t \). In our studies, we work with \( N = 100 \) firms in the portfolio.

Based on table 1, let us discuss some important points. Given the portfolio loss \( L(t) \) we can use the expression for \( L^m(t) \) to translate the portfolio loss into a tranche loss. Subsequently we can calculate the current face value of the tranche by \( F^m(t) \). Using these two quantities we can calculate the expected loss from defaults, the default leg, and the expected premium payments, the payment leg, and we can then
<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>Number of reference firms, indexed by $i$</td>
</tr>
<tr>
<td>$A$</td>
<td>Initial nominal value of contract</td>
</tr>
<tr>
<td>$A_i$</td>
<td>Nominal value of reference firm $i$, $A = \sum_{i=1}^{N} A_i$</td>
</tr>
<tr>
<td>$T$</td>
<td>Maturity of contract</td>
</tr>
<tr>
<td>$n$</td>
<td>Payment frequency, $\delta = \frac{1}{n}$</td>
</tr>
<tr>
<td>$M$</td>
<td>Number of tranches in the CDO, index by $m$</td>
</tr>
<tr>
<td>$\Gamma, \Delta$</td>
<td>Lower and upper attachment points of each tranche, $\Gamma = {\Gamma^m}<em>{m=1..M}$, $\Delta = {\Delta^m}</em>{m=1..M}$, $\Gamma^1 = 0$, $\Delta^M = A$, $\Delta^{m-1} = \Gamma^m$</td>
</tr>
<tr>
<td>$s^m$</td>
<td>Fair spread of tranche $m$</td>
</tr>
<tr>
<td>$p^m$</td>
<td>Upfront fee for tranche $m$, usually only relevant for equity tranche</td>
</tr>
<tr>
<td>$R_i$</td>
<td>Recovery in the event of default of firm $i$</td>
</tr>
<tr>
<td>$\tau_i$</td>
<td>Default time of reference firm $i$</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>Intensity process of the default time of firm $i$</td>
</tr>
<tr>
<td>$Q_i(t)$</td>
<td>Indicator of default before time $t$, $Q_i(t) = 1_{{\tau_i &lt; t}}$</td>
</tr>
<tr>
<td>$L_i$</td>
<td>Loss in default of firm $i$, $L_i = (1 - R_i) A_i$</td>
</tr>
<tr>
<td>$L(t)$</td>
<td>Total loss on reference portfolio at time $t$, $L(t) = \sum_{i=1}^{N} L_i Q_i(t)$</td>
</tr>
<tr>
<td>$L^m(t)$</td>
<td>Total loss in tranche $m$ at time $t$</td>
</tr>
<tr>
<td>$F^m(t)$</td>
<td>Face value of tranche $m$ at time $t$. $F^m(t) = (\Delta^m - \Gamma^m) - L^m(t)$.</td>
</tr>
<tr>
<td>$K$</td>
<td>Number of trials in Monte-Carlo simulation, indexed by $k$</td>
</tr>
</tbody>
</table>

Table 1: Notation used for valuation of a CDO by simulation.

calculate the fair tranche spread. More rigorously, we get for the default leg

$$DL^m = E \left[ \int_0^T D_t dL^m(t) \right]$$

and the payment leg

$$PL^m = E \left[ \sum_{i=1}^{n} s^m \delta D_{t_i} F^m(t_i) \right] + p^m F^m(0)$$

Using these two equations, we can solve for $s^m$.

As simple as it may seem, the calculations are quite complex. The $L^m$ process depends on defaults of the $N$ firms, each driven by mutually correlated intensity processes. Therefore, the two expectations involved in the expression for $s^m$ will not generally be available in closed form so we need to resort to MC. We also have to
consider that we have to find $M$ fair spreads, one for each tranche so we need an efficient implementation, which can find all fair spreads at the same time.

The algorithm is as follows.

1. Draw $N$ independent $U(0, 1)$ distributed random variable, $U_i$.

2. Simulate the $N$ intensity processes from zero to maturity of the CDO, $T$, using a suitable discretization.

3. Using equation (4) for the $U_i$’s we can determine when, and if, a particular firm defaults. Let us assume that $P$ firms default for the given path of the intensity processes. We sort the default times into a vector $\Psi = \{\tau_p\}_{k=1..P}$.

4. With the vector of default times at hand, we can calculate the value of the $L(t)$ function, and more importantly, as it changes at the default times.

5. Using the evolution of $L(t)$ we can calculate the evolution of tranche losses by the definition of $L^m(t)$ and subsequently the face value of the tranches by $F^m(t)$.

6. With the evolution of the loss and face value in each tranche we can apply equations (6) and (7) to calculate the discounted payment streams. There are two things to notice. The integral in (6) collapses to a sum of $P$ elements, one for each default. The only thing we need is the discounted changes in the loss occurring at the default times stored in the vector $\Psi$. When calculating the value of the payments in equation (7) care must be taken as the payment times $t_i$ must be mapped into the default times to accurately determine the face value at the payment time.

We repeat the simulation algorithm for the $K$ paths of the intensity processes and average the values of the tranche payment and default legs. Ultimately, we solve for the fair tranche spreads. To ensure that precision is not a problem we choose $K = 60000$ paths and we further apply antithetic sampling. To our knowledge, there is no clear guidance as to how to apply antithetic sampling when modeling default times. We choose to reverse the signs of all the shock to the Brownian Motions of the intensities. Nevertheless, we keep the $U_i$ random variables used in the definition of the default times fixed instead of taking $\hat{U}_i = 1 - U_i$ which is also a $U(0, 1)$ random variable.

With the valuation framework in place, we are ready to introduce the intensity specification used in the numerical analysis.
3.4 Intensity specification

We let the specification of the intensity of each individual firm be given as a standard square root process

\[ d\lambda_i = \kappa_i (\theta_i - \lambda_i) dt + \sigma_i \sqrt{\lambda_i} dW_{it} \]

We further assume that the instantaneous correlation between two intensity processes is given by

\[ dW_{it} dW_{jt} = \rho_{ij} dt \]

In the implementation, we initiate the intensity processes at their long run means, \( \lambda_{i0} = \theta_i \). With the assumed structure of the default intensities, correlation between default events enters through the parameters \( \rho_{ij} \).

4 Results

In section 4.1, we examine fair tranche spreads, base correlations and their dependence on the assumed intensity correlation. In doing so, we uncover that even if the intensity correlation increases, base correlations for senior tranches may actually decrease. We further show that distributional assumptions of our spread-generating model are not driving this phenomenon. In section 4.2, we move on with analyzing the relative valuation method proposed by McGinty & Ahluwalia (2004d). Here we find that the relative valuation framework may produce negative expected losses in some tranches. Finally, in section 4.3, we show that base correlations at a given attachment point depend on the placement of all prior attachment points.

Throughout the numerical examination, we assume that the recovery rate in default is \( R_i = 40\% \). We work with a fixed set of attachment points for the CDO and we use the European standard tranches, also used in the numerical example of McGinty & Ahluwalia (2004c), \( (0, 3) \), \( (3, 6) \), \( (6, 9) \), \( (9, 12) \), \( (12, 22) \) and finally \( (22, 100) \). Further, we follow McGinty & Ahluwalia (2004c) in having an interest rate of zero and finally we assume that the equity tranche receives an upfront fee of \( p^0 = 30\% \). Throughout the analysis, we work with a maturity of the CDO (and CDS) of 5 years.

We pick identical parameters for all firms, \( \kappa = 0.3 \), \( \sigma = 0.25 \) and \( \theta = 0.005 \). This implies a CDS spread of 84 bps for all firms. For valuation of CDOs, we set the correlation parameters \( \rho_{ij} \) equal to the same constant \( \rho \), which we vary in order to examine how fair spreads and base correlations behave. Throughout, we shall refer to this parameter as the intensity correlation.

With these choices, we place ourselves very close to the underlying assumptions of the standard model introduced in section 2. We assume the existence of many firms and that they all have the same CDS spread and the same correlation. This means that any problems we uncover with the base correlation framework cannot be attributed to the fact that we have made assumptions of the CDO, which are in direct contrast to the premises of the standard model. Furthermore, if the base correlation
framework turns out to be problematic with such a simplistic setup, it is likely that the same will happen when applied to real-world CDOs.

4.1 Tranche spreads and base correlations

Let us start by describing the calculated spreads and the base correlations backed out from these. Figure 1 contains two charts. The top chart shows fair spreads, by the upper attachment point, of the tranche for values of $\rho$ between zero and 0.9. The bottom chart shows the base correlations calculated for each given $\rho$. Additionally, table 2 exhibits, for reference, fair spreads and base correlations of the figures.

<table>
<thead>
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Table 2: This table contains two panels showing fair spreads and implied base correlations for varying levels of the correlation parameter $\rho$. The left panel shows fair spreads, by upper attachment point, while the right panel shows base correlations implied from the spreads of the left panel.

Focusing on the fair spreads, we find the expected relationship among tranches and across intensity correlations. For a given correlation, the tranche spreads decrease with seniority and for a given tranche, we find a monotonic relationship between fair tranche spreads and correlation. For the equity and (3,6) tranches the fair spreads decrease with increasing correlation, because correlation increases the occurrence of states with very few defaults as discussed in section 2.1. For the more senior tranches, however, tranche spreads increase with correlation since the occurrence of many defaults, which affects senior tranches, is now more likely.

Turning to the base correlations, let us fix an intensity correlation. We then have the expected relationship that the base correlation is increasing with seniority as observed in the market. Fixing instead an attachment point, we see that, for the more junior tranches, the base correlation goes up when the intensity correlation goes up, as expected. However, for the more senior tranches we find that when the intensity correlation goes up, the base correlation of the senior tranches may decrease. This means that even if spreads change as to reflect an increased correlation, the base correlations seem to imply that the correlation actually has gone down.

This is the first finding: even if the intensity correlation increases among reference entities, base correlations can decrease for senior tranches. At the outset, the motivation for introducing base correlations was to achieve a certain degree of monotonicity
Figure 1: This figure contains two charts showing fair spreads and base correlations for varying levels of $\rho$. The top chart shows, by upper attachment point, fair spreads for varying levels of $\rho$ while the bottom chart shows base correlations based on the spreads of the upper chart for each level of $\rho$. Note, that due to the shape of the surface, the directions of the axes differ in the two charts. Table 2 contains a tabulated version of the contents of the charts.

but as we uncover, this is not always the case. In turn this means that one has to be careful when stating that 'correlations decrease' when analyzing base correlations since in reality, the true model generating the data may actually have experienced an
increase in the correlation.

Based on this one could think that when the intensity correlation increases, the portfolio loss distribution is changed in an unnatural way, inducing the counterintuitive behavior of the base correlations. We now argue that this is not the case.

For this, we put forth two arguments, which we discuss in turn. First, the portfolio loss density is well-behaved: when the intensity correlation goes up, the tails thicken uniformly. Second, as observed, base correlations at for example the 12% attachment point could decrease with increased intensity correlation. However, when calculating the fair spread of a pure 12% equity tranche and calculating the base correlation from this, it will be monotonic in the intensity correlation. Thus, the lack of monotonicity is a direct consequence of the bootstrapping procedure used to extract the base correlations and not the distributional assumptions of the portfolio loss.

To establish the first point, consider figure 2. The figure shows the density of portfolio losses for varying levels of intensity correlation with the tranche attachment points indicated. It is worth noting that the expected total portfolio loss is virtually unchanged under varying correlation, at approximately 3.88% and any variation is due to uncertainty induced by the Monte Carlo method. As expected, we see that for increasing correlation the tails of the loss density become more fat. Let us focus the case of the (12, 22) tranche where we from table 2 have a consistently decreasing pattern in base correlations. From figure 2, we see that there is virtually no mass in the upper tail of the loss distribution for $\rho = 0$ while there is a distinct positive probability of losses above 12% for $\rho = 0.8$. Comparing this to table 2, we see that the base correlation at 22% drops from 54% at $\rho = 0$ to 41% at $\rho = 0.8$. This means that even if we do in fact generate a more fat upper tail in the loss distribution, induced by the increased correlation, the base correlation decreases.

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</table>

Table 3: This table shows, for varying correlations, the probability of experiencing a total portfolio loss less than or equal to a given attachment point.

An alternative way to verify that the loss distribution is well-behaved is to examine the loss distribution. For this, table 3 shows, for a range of correlations, the probability of experiencing a loss less than or equal to a given attachment point. We see that with a correlation of zero the probability of not having the equity tranche wiped out before maturity is 33.2% while it has increased to 56.5% for a correlation of 0.8. Also for the more senior tranches, there is an effect. For the (9, 12) tranche, there is virtually zero probability of any loss at a correlation of zero while the probability
increases to 9\% for a correlation of 0.8.

In summary, when focusing on the loss density and the loss distribution, we can verify that the losses behave in an intuitive fashion when the default correlation is changed.

Turning to the second point, we examine the non-monotonic relationship between the intensity correlation and the base correlation at the 12\% attachment point in closer detail. Let us focus on a pure 12\% equity tranche. If our intensity-based model is well-behaved, the fair spread will decrease when the intensity correlation goes up. Further, for equity tranches, there is a monotonic relationship between fair spread and base correlation. Thus, if the intensity-based model is well-behaved the base correlation calculated from a pure 12\% equity tranche will be monotonic in the intensity correlation. To see that this is the case, figure 3 has been included.

We show two series for varying levels of intensity correlation: the Equity series is the base correlation calculated from a theoretical 12\% equity-tranche spread while the Mezzanine series shows the base correlation calculated from the standard set of tranches. That is, the Mezzanine series is the base correlation as shown in figure 1 and table 2. As expected, we find that the base correlation calculated from a pure equity tranche is monotonically increasing with intensity correlation while the base correlation at the same attachment point, based on the tranches, is not. We will return to the somewhat suspicious difference between the two sets of base correlations in section 4.3.
We thus see that our intensity model does in fact generate base correlations, which are monotonic in the intensity correlation, but when we extract the base correlations via the bootstrapping procedure for a set of mezzanine tranches, they do not remain monotonic.

The two points above serve to underline that the problems we experience with base correlations are not caused by a counterintuitive distributional assumptions but are an artifact of the base correlation framework. It is, however, not entirely clear what the main driver behind the problem is. Based on the simple observation that the loss density integrates to one, an interpretation is that when the intensity correlation goes up and the tail thickens, some portion of the loss density inevitably becomes thinner and this is what could be picked up in the bootstrap when backing out the base correlations.

4.2 Relative valuation

As described in McGinty & Ahluwalia (2004d), we can use base correlations to value off-market tranches. This is done by linearly interpolating in the base correlations and calculating losses in equity tranches corresponding to the lower and upper attachment points of the given tranche. In this section, we evaluate this pricing methodology. The recipe is simple: with the theoretical model, we generate fair tranche spreads.
from which we use the HLPGC model to back out base correlations. Interpolating in these we can value a set of different tranches. Using the theoretical model, we can also calculate the ‘true’ spreads for these tranches, thus enabling us to examine the merits of the relative valuation tool.

We follow McGinty & Ahluwalia (2004c) in examining off-market tranches with attachment points 1%, 3%, 4%, 5%, 7%, 8%, 9%, 10%, 11% and 12%. When valuing the (0, 1) tranche we will not assume an upfront fee. We examine the valuation results under varying levels of correlation among the intensity processes as to understand how the shape of the skew affects the relative valuation.

Table 4 illustrates the result of such a relative valuation. The first panel of the table, denoted JPM spreads, denotes the fair value of tranches calculated from base correlations, by upper attachment point, for varying levels of intensity correlation. The middle panel shows the true fair spreads and finally the bottom panel shows the relative error.

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Table 4: This table shows off-market tranche spreads, by upper attachment point, calculated in the JPM setup (top panel) the actual theoretical spread (middle panel) and the relative spread errors (bottom panel).

The first thing to note is that for $\rho = 0$ we do not only have very large pricing errors. The spreads for the (11, 12) tranche is also negative. We return to this fact shortly.

For a nontrivial combination of tranches and levels of intensity correlations, we find a good fit to the actual off-market tranche spreads. A cone with relative errors
below 3%, in absolute terms, extends from the 5% attachment point at \( \rho = 0 \) to the 3% and 10% attachment points at \( \rho = 0.8 \). Outside this cone, errors tend to be much larger. Disregarding the tranche \((9,10)\) and a small wedge extending from \( \rho = 0 \) at the 4% attachment points down to the 8% attachment points at \( \rho = 0.8 \), the relative valuation framework consistently overstates the fair spread. The section with negative spread errors is a subset of the section with errors under 3% in absolute terms, so when errors are significant they are usually positive. For the \((8,9)\) tranche, we have for values of \( \rho \) of 0.2 and 0.4 and to some degree 0.6, a distinctly positive spread error, which for the subsequent \((9,10)\) tranche changes sign into a significantly negative relative error instead.

This constitutes our second finding: for some combinations of tranches and intensity correlations, the spread errors are relatively small. However, they can also be very large and for a given intensity correlation, the error can change sign from one tranche to the other. This is a cause for concern if this setup is used for relative valuation of tranches. Given a calculated fair spread, we cannot know if the error for this particular tranche is positive or negative. It also means that in this respect the base correlation framework deviates from the Black-Scholes implied volatilities where the relative pricing performance is generally believed to perform well.

Let us return to the case of the negative spread for the \((11,12)\) tranche at \( \rho = 0 \), and let us examine why this occurs. The fair spread is, as discussed in section 2, solely determined by the expected losses for equity tranches with upper attachment points equal to 11 and 12% respectively. As it turns out, the expected loss for an 11% tranche is 3.64%, using the interpolated base correlation from the original spreads of 26.6%. At the 12% attachment point, the base correlation is 29.7% and this yields an expected loss of a 12% tranche of 3.62% meaning that the expected loss in a 12% equity tranche is lower than for the 11% tranche. In turn, this causes the loss, via equation (3), to be negative.

An illustration serves to describe how this occurs. Consider figure 4, showing the expected loss for the 11% tranche for the given base correlation and how the expected loss of a 12% tranche behaves for increasing values of the base correlation. The leftmost point of the chart is for the base correlation of the 11% tranche, 26.6%. For a 12% tranche, obviously, the expected loss is higher for the same base correlation, 3.71% in this case. As theory predicts, when the correlation increases, the expected loss of the 12% tranche decreases. What happens in our case is that the base correlation of the 12% attachment point is so high that the expected loss has dropped slightly below the expected loss at the 11% attachment point. Of course, the base correlations in which we interpolate to find the value of the off-market tranches are based on positive spreads. Nevertheless, we cannot be sure that the expected losses of mezzanine tranches stay positive.

This constitutes our third finding: when facing a steep correlation skew, one may obtain negative expected losses for some tranches. One could simply argue that if this should occur, the expected loss is set to zero for this particular tranche. However, this
Figure 4: This figure illustrates how the expected loss of a 12% equity tranche evolves for varying levels of the base correlation. The expected loss of a 11% tranche for fixed base correlation of 26.6% is shown as well.

is not desirable since there is credit risk in the tranche and hence, some compensation is required. Further, a more senior tranche may have a positive expected loss so it will command a positive spread and this will cause spreads not to be monotonic in the seniority, inducing arbitrage.

As we can see, however, negative expected loss (and spreads) are rare and can only happen if the correlation skew is very steep such that a situation as depicted in figure 4 can occur. Furthermore, note that this is not an artifact of the HLPGC model, but is caused by the way base correlations are fixed for lower attachment points, such that the base correlation of the upper attachment point varies ‘freely’ and can cause the inconsistencies described above.

4.3 Uniqueness of base correlations

In this section, we show that the base correlation at a given attachment point depends on the placement of all prior attachment points. To illustrate this, we focus a specific intensity correlation, \( \rho = 0.4 \).

For the off-market tranche spreads shown in table 4 we can, using the standard model, back out the base correlations. This will then give us base correlations at attachment points 1%, 3%, 4% etc. and we can then compare this to the base correlations calculated from the spreads of tranches (0, 3), (3, 6) etc. Furthermore, based on the base correlations of the traded tranche we can interpolate to find the
base correlations at the off-market attachment points. Intuitively, the interpolated base correlations should be very close to those backed out from the off-market tranche spreads themselves.

We illustrate the experiment in figure 5 where we depict three series: the base correlations calculated from the traded tranches are denoted Traded, the interpolated base correlations are denoted Interpolated, and finally the base correlations calculated from the actual off-market spreads are denoted Off-market.

Figure 5: This figure shows three series. The first series, Traded, shows base correlations calculated from the spreads of the traded tranches. The second series, Interpolated, shows base correlations at the off-market attachment points interpolated from the traded base correlations. The third series, Off-market, shows base correlations calculated directly from the theoretical off-market tranche spreads.

What we find is extraordinary: the base correlations that we can calculate from the actual off-market tranche spreads are at a completely different level than what we obtain from interpolating in the base correlations of the original tranches. Let us focus on the 9% attachment point. We know that from the original tranche spreads the base correlation is, from table 2, 0.201. However, using the same model we calculate the spreads of the off-market tranches and from a (8, 9) tranche find, from figure 5, that the base correlation at attachment point 9% is now 0.219.

This means that the base correlation we extract appears to depend on the lower attachment points, as we already alluded to in figure 3. Let us quantify this by considering the following experiment. We create a mapping function, which for a given $x$ calculates fair spreads for attachment points $(0, x)$, $(x, 8)$ and $(8, 9)$ and
returns the base correlation of the 8 and 9% attachment points. In figure 6, we show the result where $x$ is the value on the x-axis.

Figure 6: This figure shows the base correlation of the 8% and 9% attachment points calculated from the set of tranche spreads $(0, x)$, $(x, 8)$ and $(8, 9)$ where $x$ is the value on the x-axis.

We find that the base correlations of the two attachment points depend on the choice of the prior attachment point, even if the same theoretical model generates them. For the 9% base correlation the value can be anywhere between 0.189 and 0.208 and the mere fact that it varies is discouraging since its lower attachment point, 8%, is unchanged. Nevertheless, it is affected by the fact that the other attachment point $x$ influences the base correlation at 8% and thereby the base correlation of 9%. However, for each $x$ value, we have calculated all fair spreads and evidently, the $(8, 9)$ tranche spread is the same, 1.369%, no matter what the lower attachment point. Thus, the base correlations at 8 and 9% in figure 6 all have the property that they imply the same value for the $(8, 9)$ tranche.

This constitutes our fourth and final finding: base correlations are only defined uniquely up the set of attachment points used. At a specific attachment point, the base correlation thus depends on the position of all prior attachment points. One may argue that this is a purely philosophical problem as the attachment points to use are given to us based on the liquidity in the market, but nevertheless it diminishes the strength of the analogy between base correlations and implied volatilities from equity options which of course are independent of prices of options with lower strikes.

One could also argue that the problem induced by the bootstrapping in the equity tranches exists in other places in finance and is merely a necessary evil. Consider for
example the following experiment. We focus a constant interest rate of 5% and calculate prices of discount bonds. Now assume that we only observe these discount bonds at some maturities and we bootstrap in between them. Then, if we were to try with two different sets of observed discount bonds and calculated forward rates at particular times using these different sets, then the forward rates could be different, even if the underlying curve is the same. However, the bootstrapping problem is more severe when used in the CDO context in the following sense. For the experiment with the bootstrapped yield curve, we will generally have that if the perturbation of observed times are all below a certain time \( \tau \), then all spot rates after \( \tau \) will be the same, irrespective of the perturbation of prior times. However, as figures 5 and 6 reveal, local changes in the points where a spread is observed have significant effects for all future calculated base correlations, and in the case of base correlations for traded versus off-market tranches, the two sets of base correlations are at entirely different levels.

The starting point for introducing base correlations was to improve on the compound correlations by achieving uniqueness, but as we have found above, by introducing base correlations another type of uniqueness problem appears.

Another aspect of the uniqueness problem is that, see for example Duffie (2004), the attachment points used in Europe, as experimented with here, are different from the ones used in North America, \((0, 3, 7, 10, 15, 30, 100)\). This means that, across the regions, the base correlation at a given point, for example 15\%, will have different meaning, not just because the attachment point 15\% does not exist in both regions and thus has to be interpolated, but simply because many prior attachment points are different which in itself will cause the base correlation at 15\% to be different.

5 Conclusion

In this paper, we have provided an examination of the notion of base correlations. The concept is easy to explain and due to the monotonic relationship between fair spreads and correlations of equity tranches, the method provides an improvement to the compound correlation approach in the sense that for fixed attachment points, base correlations are unique. Based on quoted fair spreads, the base correlation framework aims to provide a simple and intuitive method of quickly calculating fair spreads of off-market tranches.

We have examined how base correlations behave under changing levels of default correlation and we have examined the merits of the relative valuation tool embedded in the base correlation setup. During this analysis, we have uncovered aspects of the setup, which we can summarize as four separate findings, two related to the definition of base correlations and two points related to the relative valuation framework. We recapitulate each in turn.

First, when increasing the default correlation in the true model spreads, the base correlations can decrease for the senior tranches. This is not an artifact of a coun-
terintuitive portfolio loss distribution but can be attributed to the bootstrap method used to extract the base correlations. It further implies that statements such as 'the base correlation for the senior tranche has gone up' have to be interpreted with care and seen in relation to the movement in more junior base correlations.

Second, the base correlations are only unique up to the attachment points provided and they depend on the placement of all prior attachment points. It could be argued that this is only a philosophical problem since the traded tranches are contractually fixed. Nevertheless, it casts some light on the strength of the analogy to implied volatilities for equity options. It has the further implication that across the North American and European markets, base correlations are not directly comparable since the markets have different attachment points and therefore, by construction, will imply different base correlations, everything else equal.

Third, for a steep correlation skew, we can obtain negative expected losses in off-market tranches when using the relative valuation tool. A simple solution to this problem does not exist since more senior tranches may very well have positive expected losses. Hence, setting the expected loss to zero when it is negative will cause tranche spreads to be non-monotonic and introduce arbitrage.

Fourth, the relative error of off-market tranche spreads is quite small for certain ranges of attachment points and default correlation. However, the errors tend to be positive with the exception of some dramatic changes in sign of the error at the more senior tranches. This could make application of the relative valuation framework risky since it is not clear when the estimated tranche spread is higher or lower than the true tranche spread.

Generally, our findings indicate that even if the base correlation setup offers some benefits over the traditional compound correlation approach, the strength of the framework is still not close to that of Black-Scholes implied volatilities for equity options and base correlations should be interpreted and used with great care.
References


