A comparison of stochastic default rate models

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Abstract

For single horizon models of defaults in a portfolio, the effect of model and distribution choice on the model results is well understood. Collateralized Debt Obligations in particular have sparked interest in default models over multiple horizons. For these, however, there has been little research, and there is little understanding of the impact of various model assumptions. In this article, we investigate four approaches to multiple horizon modeling of defaults in a portfolio. We calibrate the four models to the same set of input data (average defaults and a single period correlation parameter), and examine the resulting default distributions. The differences we observe can be attributed to the model structures, and to some extent, to the choice of distributions that drive the models. Our results show a significant disparity. In the single period case, studies have concluded that when calibrated to the same first and second order information, the various models do not produce vastly different conclusions. Here, the issue of model choice is much more important, and any analysis of structures over multiple horizons should bear this in mind.

Keywords: Credit risk, default rate, collateralized debt obligations


1 Introduction

In recent years, models of defaults in a portfolio context have been well studied. Three separate approaches (CreditMetrics, CreditRisk+, and CreditPortfolioView\(^1\)) were made public in 1997. Subsequently, researchers\(^2\) have examined the mathematical structure of the various models. Each of these studies has revealed that it is possible to calibrate the models to each other and that the differences between the models lie in subtle choices of the driving distributions and in the data sources one would naturally use to feed the models.

Common to all of these models, and to the subsequent examinations thereof, is the fact that the models describe only a single period. In other words, the models describe, for a specific risk horizon, whether each asset of interest defaults within the horizon. The timing of defaults within the risk horizon is not considered, nor is the possibility of defaults beyond the horizon. This is not a flaw of the current models, but rather an indication of their genesis as approaches to risk management and capital allocation for a fixed portfolio.

Not entirely by chance, the development of portfolio models for credit risk management has coincided with an explosion in issuance of Collateralized Debt Obligations (CDO’s). The performance of a CDO structure depends on the default behavior of a pool of assets. Significantly, the dependence is not just on whether the assets default over the life of the structure, but also on when the defaults occur. Thus, while an application of the existing models can give a cursory view of the structure (by describing, for instance, the distribution of the number of assets that will default over the structure’s life), a more rigorous analysis requires a model of the timing of defaults.

In this paper, we will survey a number of extensions of the standard single-period models that allow for a treatment of default timing over longer horizons. We will examine two extensions of the CreditMetrics approach, one that models only defaults over time and a second that effectively accounts

\(^1\)See Wilson (1997).

for rating migrations. In addition, we will examine the copula function approach introduced by Li (1999 and 2000), as well as a simple version of the stochastic intensity model applied by Duffie and Garleanu (1998).

We will seek to investigate the differences in the four approaches that arise from model – rather than data – differences. Thus, we will suppose that we begin with satisfactory estimates of expected default rates over time, and of the correlation of default events over one period. Higher order information, such as the correlation of defaults in subsequent periods or the joint behavior of three or more assets, will be driven by the structure of the models. The analysis of the models will then illuminate the range of results that can arise given the same initial data. Nagpal and Bahar (1999) adopt a similar approach in the single horizon context, investigating the range of possible full distributions that can be calibrated to first and second order default statistics.

In the following section, we present terminology and notation to be used throughout. We proceed to detail the four models. Finally, we present two comparison exercises: in the first, we use closed form results to analyze default rate volatilities and conditional default probabilities, while in the second, we implement Monte Carlo simulations in order to investigate the full distribution of realized default rates.

2 Notation and terminology

In order to compare the properties of the four models, we will consider a large homogeneous pool of assets. By homogeneous, we mean that each asset has the same probability of default (first order statistics) at every time we consider; further, each pair of assets has the same joint probability of default (second order statistics) at every time.

To describe the first order statistics of the pool, we specify the cumulative default probability $q_k$ – the probability that a given asset defaults in the next $k$ years – for $k = 1, 2, \ldots T$, where $T$ is the maximum horizon we consider. Equivalently, we may specify the marginal default probability
$p_k$ – the probability that a given asset defaults in year $k$. Clearly, cumulative and marginal default probabilities are related through

$$q_k = q_{k-1} + p_k, \text{ for } k = 2, \ldots, T. \quad (1)$$

It is important to distinguish a third equivalent specification, that of *conditional default probabilities*. The conditional default probability in year $k$ is defined as the conditional probability that an asset defaults in year $k$, given that the asset has survived (that is, has not defaulted) in the first $k-1$ years. This probability is given by $p_k/(1-q_{k-1})$.

Finally, to describe the second order statistics of the pool, we specify the *joint cumulative default probability* $q_{j,k}$ – the probability that for a given pair of assets, the first asset defaults sometime in the first $j$ years and the second defaults sometime in the first $k$ years – or equivalently, the *joint marginal default probability* $p_{j,k}$ – the probability that the first asset defaults in year $j$ and the second defaults in year $k$. These two notions are related through

$$q_{j,k} = q_{j-1,k-1} + \sum_{i=1}^{j-1} p_{i,k} + \sum_{i=1}^{k-1} p_{j,i} + p_{j,k}, \text{ for } j, k = 2, \ldots, T. \quad (2)$$

In practice, it is possible to obtain first order statistics for relatively long horizons, either by observing market prices of risky debt and calibrating cumulative default probabilities as in Duffie and Singleton (1999), or by taking historical cumulative default experience from a study such as Keenan et al (2000) or Standard & Poor’s (2000). Less information is available for second order statistics, however, and therefore we will assume that we can obtain the joint default probability for the first year ($p_{1,1}$)$^3$, but not any of the joint default probabilities for subsequent years. Thus, our exercise will be to calibrate each of the four models to fixed values of $q_1, q_2, \ldots, q_T$ and $p_{1,1}$, and then to compare the higher order statistics implied by the models.

The model comparison can be a simple task of comparing values of $p_{1,2}$, $p_{2,2}$, $q_{2,2}$, and so on. However, to make the comparisons a bit more tangible, we will consider the distributions of realized

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$^3$This is a reasonable supposition, since all of the single period models mentioned previously essentially require $p_{1,1}$ as an input.
default rates. The term "default rate" is often used loosely in the literature, without a clear notion of whether default rate is synonymous with default probability, or rather is itself a random variable. To be clear, in this article, default rate is a random variable equal to the proportion of assets in a portfolio that default. For instance, if the random variable $X_i^{(k)}$ is equal to one if the $i$th asset defaults in year $k$, then the year $k$ default rate is equal to

$$\frac{1}{n} \sum_{i=1}^{n} X_i^{(k)}. \tag{3}$$

For our homogeneous portfolio, the mean year $k$ default rate is simply $p_k$, the marginal default probability for year $k$. Furthermore, the standard deviation of the year $k$ default rate (which we will refer to as the *year $k$ default rate volatility*) is

$$\sqrt{p_{k,k} - p_k^2 + (p_k - p_{k,k})/n}. \tag{4}$$

Of interest to us is the large portfolio limit (that is, $n \to \infty$) of this quantity, normalized by the default probability. We will refer to this as the *normalized year $k$ default volatility*, which is given by

$$\frac{\sqrt{p_{k,k} - p_k^2}}{p_k}. \tag{5}$$

Additionally, we will examine the *normalized cumulative year $k$ default volatility*, which is defined similarly to the above, with the exception that the default rate is computed over the first $k$ years rather than year $k$ only. The normalized cumulative default volatility is given by

$$\frac{\sqrt{q_{k,k} - q_k^2}}{q_k}. \tag{6}$$

Finally, we will use $\Phi$ to denote the standard normal cumulative distribution function. In the bivariate setting, we will use $\Phi_2(z_1, z_2; \rho)$ to indicate the probability that $Z_1 < z_1$ and $Z_2 < z_2$, where $Z_1$ and $Z_2$ are standard normal random variables with correlation $\rho$.

In the following four sections, we describe the models to be considered, and discuss in detail the calibration to our initial data.
3 Discrete CreditMetrics extension

In its simplest form, the single period CreditMetrics model, calibrated for our homogeneous portfolio, can be stated as follows:

(i) Define a default threshold \( \alpha \) such that \( \Phi(\alpha) = p_1 \).

(ii) To each asset \( i \), assign a standard normal random variable \( Z_i^{(1)} \), where the correlation between distinct \( Z_i^{(1)} \) and \( Z_j^{(1)} \) is equal to \( \rho \), such that

\[
\Phi_2(\alpha, \alpha; \rho) = p_{1,1}.
\]

(iii) Asset \( i \) defaults in year 1 if \( Z_i^{(1)} < \alpha \).

The simplest extension of this model to multiple horizons is to simply repeat the one period model. We then have default thresholds \( \alpha_1, \alpha_2, \ldots, \alpha_T \) corresponding to each period. For the first period, we assign standard normal random variables \( Z_i^{(1)} \) to each asset as above, and asset \( i \) defaults in the first period if \( Z_i^{(1)} < \alpha_1 \). For assets that survive the first period, we assign a second set of standard normal random variables \( Z_i^{(2)} \), such that the correlation between distinct \( Z_i^{(2)} \) and \( Z_j^{(2)} \) is \( \rho \) but the variables from one period to the next are independent. Asset \( i \) then defaults in the second period if \( Z_i^{(1)} > \alpha_1 \) (it survives the first period) and \( Z_i^{(2)} < \alpha_2 \). The extension to subsequent periods should be clear. In the end, the model is specified by the default thresholds \( \alpha_1, \alpha_2, \ldots, \alpha_T \) and the correlation parameter \( \rho \).

To calibrate this model to our cumulative default probabilities \( q_1, q_2, \ldots, q_T \) and joint default probability, we begin by setting the first period default threshold:

\[
\alpha_1 = \Phi^{-1}(q_1).
\]

For subsequent periods, we set \( \alpha_k \) such that the probability that \( Z_i^{(k)} < \alpha_k \) is equal to the conditional default probability for period \( k \):

\[
\alpha_k = \Phi^{-1}\left(\frac{q_k - q_{k-1}}{1 - q_{k-1}}\right).
\]
We complete the calibration by choosing $\rho$ to satisfy (7), with $\alpha$ replaced by $\alpha_1$.

The joint default probabilities and default volatilities are easily obtained in this context. For instance, the marginal year two joint default probability is given by (for distinct $i$ and $j$):

$$p_{2,2} = P \left\{ Z_1^{(i)} > \alpha_1 \cap Z_1^{(j)} > \alpha_1 \cap Z_2^{(i)} < \alpha_2 \cap Z_2^{(j)} < \alpha_2 \right\}$$

$$= P \left\{ Z_1^{(i)} > \alpha_1 \cap Z_1^{(j)} > \alpha_1 \right\} \cdot P \left\{ Z_2^{(i)} < \alpha_2 \cap Z_2^{(j)} < \alpha_2 \right\}$$

$$= (1 - 2p_1 + p_{1,1}) \cdot \Phi_2(\alpha_2, \alpha_2; \rho).$$

(10)

Similarly, the probability that asset $i$ defaults in the first period, and asset $j$ in the second period is

$$p_{1,2} = P \left\{ Z_1^{(i)} < \alpha_1 \cap Z_1^{(j)} > \alpha_1 \cap Z_2^{(j)} < \alpha_2 \right\} = (p_1 - p_{1,1}) \cdot \frac{q_2 - p_1}{1 - p_1}.$$  (11)

It is then possible to obtain $q_{2,2}$ using (2) and the default volatilities using (5) and (6).

4 Diffusion-driven CreditMetrics extension

By construction, the discrete CreditMetrics extension above does not allow for any correlation of default rates through time. For instance, if a high default rate is realized in the first period, this has no bearing on the default rate in the second period, since the default drivers for the second period (the $Z_2^{(i)}$ above) are independent of the default drivers for the first. Intuitively, we would not expect this behavior from the market. If a high default rate occurs in one period, then it is likely that those obligors that did not default would have generally decreased in credit quality. The impact would then be that the default rate for the second period would also have a tendency to be high.

In order to capture this behavior, we introduce a CreditMetrics extension where defaults in consecutive periods are not driven by independent random variables, but rather by a single diffusion process. Our diffusion-driven CreditMetrics extension is described by:

(i) Define default thresholds $\alpha_1, \alpha_2, \ldots, \alpha_T$ for each period.
(ii) To each obligor, assign a standard Wiener process $W^{(i)}$, with $W^{(i)}_0 = 0$, where the instantaneous correlation between distinct $W^{(i)}$ and $W^{(j)}$ is $\rho$.\(^4\)

(iii) Obligor $i$ defaults in the first year if $W^{(i)}_1 < \alpha_1$.

(iv) For $k > 1$, obligor $i$ defaults in year $k$ if it survives the first $k - 1$ years (that is, $W^{(i)}_1 > \alpha_1, \ldots, W^{(i)}_{k-1} > \alpha_{k-1}$) and $W^{(i)}_k < \alpha_k$.

Note that this approach allows for the behavior mentioned above. If the default rate is high in the first year, this is because many of the Wiener processes have fallen below the threshold $\alpha_1$. The Wiener processes for non-defaulting obligors will have generally trended downward as well, since all of the Wiener processes are correlated. This implies a greater likelihood of a high number of defaults in the second year. In effect, then, this approach introduces a notion of credit migration. Cases where the Wiener process trends downward but does not cross the default threshold can be thought of as downgrades, while cases where the process trends upward are essentially upgrades.

To calibrate the first threshold $\alpha_1$, we observe that

$$P \left\{ W^{(i)}_1 < \alpha_1 \right\} = \Phi(\alpha_1),$$

and thus that $\alpha_1$ is given by (8). For the second threshold, we require that the probability that an obligor defaults in year two is equal to $p_2$:

$$P \left\{ W^{(i)}_1 > \alpha_1 \cap W^{(i)}_2 < \alpha_2 \right\} = p_2.$$  \hfill (13)

Since $W^{(i)}$ is a Wiener process, we know that the standard deviation of $W^{(i)}_t$ is $\sqrt{t}$ and that for $s < t$, the correlation between $W^{(i)}_s$ and $W^{(i)}_t$ is $\sqrt{s/t}$. Thus, given $\alpha_1$, we find the value of $\alpha_2$ that satisfies

$$\Phi(\alpha_2/\sqrt{2}) - \Phi_2(\alpha_1, \alpha_2/\sqrt{2}; \sqrt{1/2}) = p_2.$$  \hfill (14)

\(^4\)Technically, the cross variation process for $W^{(i)}$ and $W^{(j)}$ is $\rho dt$.  

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For the $k$th period, given $\alpha_1, \ldots, \alpha_{k-1}$, we calibrate $\alpha_k$ by solving
\[
P \left\{ W^{(i)}_1 > \alpha_1 \cap \ldots \cap W^{(i)}_{k-1} > \alpha_{k-1} \cap W^{(i)}_k < \alpha_k \right\} = p_k,
\]
again utilizing the properties of the Wiener process $W^{(i)}$ to compute the probability on the left hand side.

We complete the calibration by finding $\rho$ such that the year one joint default probability is $p_{1,1}$:
\[
P \left\{ W^{(i)}_1 < \alpha_1 \cap W^{(j)}_1 < \alpha_1 \right\} = p_{1,1}.
\]
Since $W^{(i)}_1$ and $W^{(j)}_1$ each follow a standard normal distribution, and have a correlation of $\rho$, the solution for $\rho$ here is identical to that of the previous section.

With the calibration complete, it is a simple task to compute the joint default probabilities. For instance, the joint year two default probability is given by
\[
p_{2,2} = P \left\{ W^{(i)}_1 > \alpha_1 \cap W^{(j)}_1 > \alpha_1 \cap W^{(i)}_2 < \alpha_2 \cap W^{(j)}_2 < \alpha_2 \right\},
\]
where we use the fact that $\{W^{(i)}_1, W^{(j)}_1, W^{(i)}_2, W^{(j)}_2\}$ follow a multivariate normal distribution with covariance
\[
\text{Cov}\{W^{(i)}_1, W^{(j)}_1, W^{(i)}_2, W^{(j)}_2\} = \begin{pmatrix}
1 & \rho & 1 & \rho \\
\rho & 1 & \rho & 1 \\
1 & \rho & 2 & 2\rho \\
\rho & 1 & 2\rho & 2
\end{pmatrix}.
\]

5 Copula functions

A drawback of both the CreditMetrics extensions above is that in a Monte Carlo setting, they require a stepwise simulation approach. In other words, we must simulate the pool of assets over the first year, tabulate the ones that default, then simulate the remaining assets over the second year, and so on. Li (1999 and 2000) introduces an approach wherein it is possible to simulate the default times directly, thus avoiding the need to simulate each period individually.

The normal copula function approach is as follows:
(i) Specify the cumulative default time distribution $F$, such that $F(t)$ gives the probability that a given asset defaults prior to time $t$.

(ii) Assign a standard normal random variable $Z^{(i)}$ to each asset, where the correlation between distinct $Z^{(i)}$ and $Z^{(j)}$ is $\rho$.

(iii) Obtain the default time $\tau_i$ for asset $i$ through

$$\tau_i = F^{-1}(\Phi(Z^{(i)})).$$  \hspace{1cm} (19)

Since we are concerned here only with the year in which an asset defaults, and not the precise timing within the year, we will consider a discrete version of the copula approach:

(i) Specify the cumulative default probabilities $q_1, q_2, \ldots, q_T$ as in Section 2.

(ii) For $k = 1, \ldots, T$ compute the threshold $\alpha_k = \Phi^{-1}(q_k)$. Clearly, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_T$. Define $\alpha_0 = -\infty$.

(iii) Assign $Z^{(i)}$ to each asset as above.

(iv) Asset $i$ defaults in year $k$ if $\alpha_{k-1} < Z^{(i)} \leq \alpha_k$.

The calibration to the cumulative default probabilities is already given. Further, it is easy to observe\(^5\) that the correlation parameter $\rho$ is calibrated exactly as in the previous two sections.

The joint default probabilities are perhaps simplest to obtain for this approach. For example, the joint cumulative default probability $q_{k,l}$ is given by

$$q_{k,l} = P \left\{ Z^{(i)} < \alpha_k \cap Z^{(j)} < \alpha_l \right\} = \Phi_2(\alpha_k, \alpha_l; \rho).$$  \hspace{1cm} (20)

\(^5\)Details are presented in Li (1999) and Li (2000).
6  Stochastic default intensity

6.1  Description of the model

The approaches of the three previous sections can all be thought of as extensions of the single period CreditMetrics framework. Each approach relies on standard normal random variables to drive defaults, and calibrates thresholds for these variables. Furthermore, it is easy to see that over the first period, the three approaches are identical; they only differ in their behavior over multiple periods.

Our fourth model takes a different approach to the construction of correlated defaults over time, and can be thought of as an extension of the single period CreditRisk+ framework. In the CreditRisk+ model, correlations between default events are constructed through the assets’ dependence on a common default probability, which itself is a random variable. \(^6\) Importantly, given the realization of the default probability, defaults are conditionally independent. The volatility of the common default probability is in effect the correlation parameter for this model; a higher default volatility induces stronger correlations, while a zero volatility produces independent defaults. \(^7\)

The natural extension of the CreditRisk+ framework to continuous time is the stochastic intensity approach presented in Duffie and Garleanu (1998) and Duffie and Singleton (1999). Intuitively, the stochastic intensity model stipulates that in a given small time interval, assets default independently, with probability proportional to a common default intensity. \(^8\) In the next time interval, the intensity changes, and defaults are once again independent, but with the default probability proportional to the new intensity level. The evolution of the intensity is described through a stochastic process. In practice, since the intensity must remain positive, it is common to apply similar stochastic processes as are utilized in models of interest rates.

\(^6\) More precisely, assets may depend on different default probabilities, each of which are correlated.

\(^7\) See Finger (1998), Gordy (2000), and Kolyoglu and Hickman (1998) for further discussion.

\(^8\) As with our description of the CreditRisk+ model, this is a simplification. The Duffie-Garleanu framework provides for an intensity process for each asset, with the processes being correlated.
For our purposes, we will model a single intensity process \( h \). Conditional on \( h \), the default time for each asset is then the first arrival of a Poisson process with arrival rate given by \( h \). The Poisson processes driving the defaults for distinct assets are independent, meaning that given a realization of the intensity process \( h \), defaults are independent. The Poisson process framework implies that given \( h \), the probability that a given asset survives until time \( t \) is

\[
\exp \left[ -\int_0^t du \, h_u \right].
\]

Further, because defaults are conditionally independent, the conditional probability, given \( h \), that two assets both survive until time \( t \) is

\[
\exp \left[ -2 \int_0^t du \, h_u \right].
\]

The unconditional survival probabilities are given by expectations over the process \( h \), so that in particular, the survival probability for a single asset is given by

\[
1 - q_t = \mathbb{E} \exp \left[ -\int_0^t du \, h_u \right].
\]

For the intensity process, we assume that \( h \) evolves according to the stochastic differential equation

\[
d h_t = -\kappa (h_t - \bar{h}_k) dt + \sigma \sqrt{h_t} dW_t,
\]

where \( W \) is a Wiener process and \( \bar{h}_k \) is the level to which the process trends during year \( k \). (That is, the mean reversion is toward \( \bar{h}_1 \) for \( t < 1 \), toward \( \bar{h}_2 \) for \( 1 \leq t < 2 \), etc.) Let \( h_0 = \bar{h}_1 \). Note that this is essentially the model for the instantaneous discount rate used in the Cox-Ingersoll-Ross interest rate model. Note also that in Duffie-Garleanu, there is a jump component to the evolution of \( h \), while the level of mean reversion is constant.

In order to express the default probabilities implied by the stochastic intensity model in closed form, we will rely on the following result from Duffie-Garleanu.\(^9\) For a process \( h \) with \( h_0 = \bar{h} \) and

\(^9\)We have changed the notation slightly from the Duffie-Garleanu result, in order to make more explicit the dependence on \( \bar{h} \).
evolving according to (24) with $\tilde{h}_k = \tilde{h}$ for all $k$, we have

$$E_t \exp \left[ - \int_t^{t+s} du \, h_u \right] \exp[x + y h_s] = \exp \left[ x + \alpha_s(y) \tilde{h} + \beta_s(y) h_t \right],$$

where $E_t$ denotes conditional expectation given information available at time $t$. The functions $\alpha_s$ and $\beta_s$ are given by

$$\alpha_s(y) = \frac{\kappa}{c} + \frac{\kappa(a(y)c - d(y))}{bcd(y)} \log \left[ \frac{c + d(y)e^{bs}}{c + d} \right],$$

$$\beta_s(y) = \frac{1 + a(y)e^{bs}}{c + d(y)e^{bs}},$$

where

$$c = -\frac{\kappa + \sqrt{\kappa^2 + 2\sigma^2}}{2},$$

$$d(y) = (1 - cy) \frac{\sigma^2 y - \kappa + \sqrt{(\sigma^2 y - \kappa)^2 - \sigma^2 (\sigma^2 y^2 - 2\kappa y - 2)}}{\sigma^2 y^2 - 2\kappa y - 2},$$

$$a(y) = (d(y) + c)y - 1,$$

$$b = \frac{-d(y)(\kappa + 2c) + a(y)(\sigma^2 - \kappa c)}{a(y)c - d(y)}.$$

### 6.2 Calibration

Our calibration approach for this model will be to fix the mean reversion speed $\kappa$, solve for $\tilde{h}_1$ and $\sigma$ to match $p_1$ and $p_{1,1}$, and then to solve in turn for $\tilde{h}_2, \ldots, \tilde{h}_T$ to match $p_2, \ldots, p_T$. To begin, we apply (23) and (25) to obtain

$$p_1 = 1 - \exp \left[ \alpha_1(0) \tilde{h}_1 + \beta_1(0) h_0 \right] = 1 - \exp \left[ (\alpha_1(0) + \beta_1(0)) \tilde{h}_1 \right].$$

To compute the joint probability that two obligors each survive the first year, we must take the expectation of (22), which is essentially the same computation as above, but with the process $h$ replaced by $2h$. We observe that the process $2h$ also evolves according to (24) with the same mean reversion speed $\kappa$, and with $\tilde{h}_k$ replaced by $2\tilde{h}_k$ and $\sigma$ replaced by $\sigma \sqrt{2}$. Thus, we define the
functions $\hat{\alpha}_s$ and $\hat{\beta}_s$ in the same way as $\alpha_s$ and $\beta_s$, with $\sigma$ replaced by $\sigma\sqrt{2}$. We can then compute the joint one year survival probability:

$$\mathbb{E}\exp\left[-2\int_0^t du \, h_u\right] = \exp\left[2(\hat{\alpha}_1(0) + \hat{\beta}_1(0))\tilde{h}_1\right].$$  \hfill (33)

Finally, since the joint survival probability is equal to $1 - 2p_1 + p_{1,1}$, we have

$$p_{1,1} = 2p_1 - 1 + \exp\left[2(\hat{\alpha}_1(0) + \hat{\beta}_1(0))\tilde{h}_1\right].$$  \hfill (34)

To calibrate $\sigma$ and $\tilde{h}_1$ to (32) and (34), we first find the value of $\sigma$ such that

$$\frac{2(\hat{\alpha}_1(0) + \hat{\beta}_1(0))}{\alpha_1(0) + \beta_1(0)} = \frac{\log[1 - 2p_1 + p_{1,1}]}{\log[1 - p_1]},$$  \hfill (35)

and then set

$$\tilde{h}_1 = \frac{\log[1 - p_1]}{\alpha_1(0) + \beta_1(0)}.$$  \hfill (36)

Note that though the equations are lengthy, the calibration is actually quite straightforward, in that we only are ever required to fit one parameter at a time.

In order to calibrate $\tilde{h}_2$, we need to obtain an expression for the two year cumulative default probability $q_2$. To this end, we must compute the two year survival probability

$$1 - q_2 = \mathbb{E}\exp\left[-\int_0^2 du \, h_u\right].$$  \hfill (37)

Since the process $h$ does not have a constant level of mean reversion over the first two years, we cannot apply (25) directly here. However (25) can be applied once we express the two year survival probability as

$$1 - q_2 = \mathbb{E}\exp\left[-\int_0^1 du \, h_u\right] \mathbb{E}_1\exp\left[-\int_1^2 du \, h_u\right].$$  \hfill (38)

Now given $h_1$, the process $h$ evolves according to (24) from $t = 1$ to $t = 2$ with a constant mean reversion level $\tilde{h}_2$, meaning we can apply (25) to the conditional expectation in (38), yielding

$$1 - q_2 = \mathbb{E}\exp\left[-\int_0^1 du \, h_u\right] \exp[\alpha_1(0)\tilde{h}_2 + \beta_1(0)h_1].$$  \hfill (39)
The same argument allows us to apply (25) again to (39), giving

\[ 1 - q_2 = \exp \left[ \alpha_1(0) \tilde{h}_2 + \left( \alpha_1(0) + \beta_1(0) \right) \tilde{h}_1 \right]. \]  
\[ (40) \]

Thus, our calibration for the second year requires setting

\[ \tilde{h}_2 = \frac{1}{\alpha_1(0)} \left\{ \log \left[ 1 - q_2 \right] - \left( \alpha_1(0) + \beta_1(0) \right) \tilde{h}_1 \right\}. \]  
\[ (41) \]

The remaining mean reversion levels \( \tilde{h}_3, \ldots, \tilde{h}_T \) are calibrated similarly.

### 6.3 Joint default probabilities

The computation of joint probabilities for longer horizons is similar to (34). The joint probability that two obligors each survive the first two years is given by

\[ \mathbb{E} \exp \left[ -2 \int_0^2 du \, h_u \right]. \]  
\[ (42) \]

Here, we apply the same arguments as in (38) through (40) to derive

\[ \mathbb{E} \exp \left[ -2 \int_0^2 du \, h_u \right] = \exp \left[ 2\hat{\alpha}_1(0) \tilde{h}_2 + 2[\hat{\alpha}_1(0) + \hat{\beta}_1(0)] \tilde{h}_1 \right]. \]  
\[ (43) \]

For the joint probability that the first obligor survives the first year and the second survives the first two years, we must compute

\[ \mathbb{E} \exp \left[ - \int_0^1 du \, h_u \right] \exp \left[ -\int_0^2 du \, h_u \right] = \mathbb{E} \exp \left[ -2 \int_0^1 du \, h_u \right] \exp \left[ -\int_1^2 du \, h_u \right] \]  
\[ (44) \]

The same reasoning yields

\[ \mathbb{E} \exp \left[ - \int_0^1 du \, h_u \right] \exp \left[ -\int_0^2 du \, h_u \right] = \exp \left[ \alpha_1(0) \tilde{h}_2 + 2[\hat{\alpha}_1(0)/2 + \hat{\beta}_1(0)/2] \tilde{h}_1 \right]. \]  
\[ (45) \]

The joint default probabilities \( p_{2,2} \) and \( p_{1,2} \) then follow from (43) and (45).
7 Model comparisons – closed form results

Our first set of model comparisons will utilize the closed form results described in the previous sections. We will restrict the comparisons here to the two period setting, and to second order results (that is, default volatilities and joint probabilities for two assets); results for multiple periods and actual distributions of default rates will be analyzed through Monte Carlo in the next section.

For our two period comparisons, we will analyze four sets of parameters: investment and speculative grade default probabilities\(^{10}\), each with two correlation values. The low and high correlation settings will correspond to values of 10% and 40%, respectively, for the asset correlation parameter \(\rho\) in the first three models. For the stochastic intensity model, we will investigate two values for the mean reversion speed \(\kappa\). The "slow" setting will correspond to \(\kappa = 0.29\), such that a random shock to the intensity process will decay by 25% over the next year; the "fast" setting will correspond to \(\kappa = 1.39\), such that a random shock to the intensity process will decay by 75% over one year.

Calibration results are presented in Table 1.

We present the normalized year two default volatilities for each model in Figure 1. As defined in (5) and (6), the marginal and cumulative default volatilities are the standard deviation of the marginal and cumulative two year default rates of a large, homogeneous portfolio. As we would expect, the default volatilities are greater in the high correlation cases than in the low correlation cases. Of the five models tested, the stochastic intensity model with slow mean reversion seems to produce the highest levels of default volatility, indicating that correlations in the second period tend to be higher for this model than for the others.

It is interesting to note that of the first three models, all of which are based on the normal distribution and default thresholds, the copula approach in all four cases has a relatively low marginal default volatility but a relatively high cumulative default volatility. (The slow stochastic intensity model is in fact the only other model to show a marginal volatility less than the cumulative volatility.) Note

\(^{10}\)Taken from Exhibit 30 of Keenan et al (2000).
that the cumulative two year default rate is the sum of the first and second year marginal default rates, and thus that the two year cumulative default volatility is composed of three terms: the first and second year marginal default volatilities and the covariance between the first and second years. Our calibration guarantees that the first year default volatilities are identical across the models. Thus, the behavior of the copula model suggests a stronger covariance term (that is, a stronger link between year one and year two defaults) than for either of the two CreditMetrics extensions.

To further investigate the links between default events, we examine conditional probability of a default in the second year, given the default of another asset. To be precise, for two distinct assets $i$ and $j$, we will calculate the conditional probability that asset $i$ defaults in year two, given that asset $j$ defaults in year one, normalized by the unconditional probability that asset $i$ defaults in year two. In terms of quantities we have already defined, this normalized conditional probability is equal to $p_{1;2} = p_{1} p_{2}/p_{1;2}$. We will also calculate the normalized conditional probability that asset $i$ defaults in year two, given that asset $j$ defaults in year two, denoted by $p_{2;2} = p_{2}/p_{2;2}$. For both of these quantities, a value of one indicates that the first asset defaulting does not affect the chance that the second asset defaults; a value of four indicates that the second asset is four times more likely to default if the first asset defaults than it is if we have no information about the first asset. Thus, the probability conditional on a year two default can be interpreted as an indicator of contemporaneous correlation of defaults, and the probability conditional on a year one default as an indicator of lagged default correlation.

The normalized conditional probabilities under the five models are presented in Figure 2. As we expect, there is no lagged correlation for the discrete CreditMetrics extension. Interestingly, the copula and both stochastic intensity models often show a higher lagged than contemporaneous correlation. While it is difficult to establish much intuition for the copula model, this phenomenon can be rationalized in the stochastic intensity setting. For this model, any shock to the default intensity will tend to persist longer than one year. If one asset defaults in the first year, it is most likely due to a positive shock to the intensity process; this shock then persists into the second year, where the other asset is more likely to default than normal. Further, shocks are more persistent for the
slower mean reversion, explaining why the difference in lagged and contemporaneous correlation is more pronounced in this case. By contrast, the two CreditMetrics extensions show much higher contemporaneous than lagged correlation; this lack of persistence in the correlation structure will manifest itself more strongly over longer horizons.

To this point, we have calibrated the collection of models to have the same means over two periods, and the same volatilities over one period. We have then investigated the remaining second order statistics – the second period volatility and the correlation between the first and second periods – that depend on the particular models. In the next section, we will extend the analysis on two fronts: first, we will investigate more horizons in order to examine the effects of lagged and contemporaneous correlations over longer times; second, we will investigate the entire distribution of portfolio defaults rather than just the second order moments.

8 Model comparisons – simulation results

In this section, we perform Monte Carlo simulations for the five models investigated previously. In each case, we begin with a homogeneous portfolio of one hundred speculative grade bonds. We calibrate the model to the cumulative default probabilities in Table 2 and to the two correlation settings from the previous section. Over 1,000 trials, we simulate the number of bonds that default within each year, up to a final horizon of six years.11

The simulation procedures are straightforward for the two CreditMetrics extensions and the copula approach. For the stochastic intensity framework, we simulate the evolution of the intensity process according to (24). This requires a discretization of (24):

$$h_{t+\Delta t} \approx -\kappa (h_t - \bar{h}_k) \Delta t + \sigma \sqrt{h_t} \sqrt{\Delta t} \epsilon,$$

(46)

11 As we have pointed out before, it is possible to simulate continuous default times under the copula and stochastic intensity frameworks. In order to compare with the two CreditMetrics extensions, we restrict the analysis to annual buckets.
where $\epsilon$ is a standard normal random variable.\footnote{Note that while (24) guarantees a non-negative solution for $h$, the discretized version admits a small probability that $h_t + \Delta t$ will be negative. To reduce this possibility, we choose $\Delta t$ for each timestep such that the probability that $h_t + \Delta t < 0$ is sufficiently small. The result is that while we only need 50 timesteps per year in some cases, we require as many as one thousand when the value of $\sigma$ is large, as in the high correlation, fast mean reversion case.} Given the intensity process path for a particular scenario, we then compute the conditional survival probability for each annual period as in (21). Finally, we generate defaults by drawing independent binomial random variables with the appropriate probability.

The simulation time for the five models is a direct result of the number of timesteps needed. The copula model simulates the default times directly, and is therefore the fastest. The two CreditMetrics models require only annual timesteps, and require roughly 50\% more runtime than the copula model. For the stochastic intensity model, the need to simulate over many timesteps produces a runtime over one hundred times greater than the simpler models.

We first examine default rate volatilities over the six horizons. As in the previous section, we consider the normalized cumulative default rate volatility. For year $k$, this is the standard deviation of the number of defaults that occur in years one through $k$, divided by the expected number of defaults in that period. This is essentially the quantity defined in (6), with the exception that here we consider a finite portfolio. The default volatilities from our simulations are presented in Figure 3. Our calibration guarantees that the first year default volatilities are essentially the same. The second year results are similar to those in Figure 1, with slightly higher volatility for the slow stochastic intensity model, and slightly lower volatility for the discrete CreditMetrics extension. At longer horizons, these differences are amplified: the slow stochastic intensity and discrete CreditMetrics models show high and low volatilities, respectively, while the remaining three models are indistinguishable.

Though default rate volatilities are illustrative, they do not provide us information about the full distribution of defaults through time. At the one year horizon, our calibration guarantees that volatility will be consistent across the five models; the distribution assumptions, however influence the pre-
cise shape of the portfolio distribution. We see in Table 3 that there is actually very little difference between even the 1st percentiles of the distributions, particularly in the low correlation case. For the full six year horizon, Table 4 shows more differences between the percentiles. Consistent with the default volatility results, the tail percentiles are most extreme for the slow stochastic intensity model, and least extreme for discrete CreditMetrics. Interestingly, though the CreditMetrics diffusion model shows similar volatility to the copula and fast stochastic intensity models, it produces less extreme percentiles than these other models. Note also that among distributions with similar means, the median serves well as an indicator of skewness. The high correlation setting generally, and the slow stochastic intensity model in particular, show lower medians. For these cases, the distribution places higher probability on the worst default scenarios as well as the scenarios with few or no defaults.

The cumulative probability distributions for the six year horizons are presented in Figures 4 through 7. As in the other comparisons, the slow stochastic intensity model is notable for placing large probability on the very low and high default rate scenarios, while the discrete CreditMetrics extension stands out as the most benign of the distributions. Most striking, however, is the similarity between the fast stochastic intensity and copula models, which are difficult to differentiate even at the most extreme percentile levels.

As a final comparison of the default distributions, we consider the pricing of a simple structure written on our portfolio. Suppose each of the one hundred bonds in the portfolio has a notional value of $1 million, and that in the event of a default the recovery rate on each bond is forty percent. The structure is composed of three elements:

(i) **First loss protection.** As defaults occur, the protection seller reimburses the structure up to a total payment of $10 million. Thus, the seller pays $600,000 at the time of the first default, $600,000 at the time of each of the subsequent fifteen defaults, and $400,000 at the time of the seventeenth default.

(ii) **Second loss protection.** The protection seller reimburses the structure for losses in excess of
$10 million, up to a total payment of $20 million. This amounts to reimbursing the losses on the seventeenth through the fiftieth defaults.

(iii) Senior notes. Notes with a notional value of $100 million maturing after six years. The notes suffer a principal loss if the first and second loss protection are fully utilized – that is, if more than fifty defaults occur.

For the first and second loss protection, we will estimate the cost of the protection based on a constant discount rate of 7%. In each scenario, we produce the timing and amounts of the protection payments, and discount these back to the present time. The price of the protection is then the average discounted value across the 1,000 scenarios. For the senior notes, we compute the expected principal loss at maturity, which is used by Moody’s along with Table 5 to determine the notes’ rating. Additionally, we compute the total amount of protection (capital) required to achieve a rating of A3 (an expected loss of 0.5%) and Aa3 (an expected loss of 0.101%).

We present the first and second loss prices in Table 6, along with the expected loss, current rating, and required capital for the senior notes. The slow stochastic intensity model yields the lowest pricing for the first loss protection, the worst rating for the senior notes, and the highest required capital. The results for the other models are as expected, with the copula and fast mean reversion models yielding the most similar results.

9 Conclusion

The analysis of Collateralized Debt Obligations, and other structured products written on credit portfolios, requires a model of correlated defaults over multiple horizons. For single horizon models, the effect of model and distribution choice on the model results is well understood. For the multiple horizon models, however, there has been little research.

We have outlined four approaches to multiple horizon modeling of defaults in a portfolio. We have calibrated the four models to the same set of input data (average defaults and a single period
correlation parameter), and have investigated the resulting default distributions. The differences we observe can be attributed to the model structures, and to some extent, to the choice of distributions that drive the models. Our results show a significant disparity. The rating on a class of senior notes under our low correlation assumption varied from Aaa to A3, and under our high correlation assumption from A1 to Baa3. Additionally, the capital required to achieve a target investment grade rating varied by as much as a factor of two.

In the single period case, a number of studies have concluded that when calibrated to the same first and second order information, the various models do not produce vastly different conclusions. Here, the issue of model choice is much more important, and any analysis of structures over multiple horizons should heed this potential model error.

References


http://www.stanford.edu/~duffie/working.htm

http://www.stanford.edu/~duffie/working.htm

http://www.riskmetrics.com/research/working


http://www.riskmetrics.com/research/techdoc

http://www.riskmetrics.com/research/journals


http://www.riskmetrics.com/research/journals


Table 1: Calibration results.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Investment grade</th>
<th>Speculative grade</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low correlation</td>
<td>High correlation</td>
</tr>
<tr>
<td></td>
<td>p₁</td>
<td>p₁</td>
</tr>
<tr>
<td></td>
<td>0.16%</td>
<td>0.16%</td>
</tr>
<tr>
<td></td>
<td>p₂</td>
<td>p₂</td>
</tr>
<tr>
<td></td>
<td>0.33%</td>
<td>0.33%</td>
</tr>
<tr>
<td></td>
<td>p₁,₁</td>
<td>p₁,₁</td>
</tr>
<tr>
<td></td>
<td>0.0007%</td>
<td>0.0059%</td>
</tr>
</tbody>
</table>

Discrete CreditMetrics extension

| α₁       | -2.95           | -2.95             | -1.83           | -1.83             |
| α₂       | -2.72           | -2.72             | -1.81           | -1.81             |
| ρ        | 10%             | 40%               | 10%             | 40%               |

Diffusion CreditMetrics extension

| α₁       | -2.95           | -2.95             | -1.83           | -1.83             |
| α₂       | -3.78           | -3.78             | -2.34           | -2.34             |
| ρ        | 10%             | 40%               | 10%             | 40%               |

Copula functions

| α₁       | -2.95           | -2.95             | -1.83           | -1.83             |
| α₂       | -2.58           | -2.58             | -1.49           | -1.49             |
| ρ        | 10%             | 40%               | 10%             | 40%               |

Stochastic intensity – slow mean reversion

| κ        | 0.29            | 0.29              | 0.29            | 0.29              |
| σ        | 0.10            | 0.37              | 0.28            | 0.76              |
| h₁       | 0.16%           | 0.16%             | 3.44%           | 3.67%             |
| h₂       | 1.47%           | 1.58%             | 6.06%           | 12.10%            |

Stochastic intensity – fast mean reversion

| κ        | 1.39            | 1.39              | 1.39            | 1.39              |
| σ        | 0.14            | 0.53              | 0.40            | 1.12              |
| h₁       | 0.16%           | 0.16%             | 3.44%           | 3.68%             |
| h₂       | 0.53%           | 0.55%             | 4.00%           | 5.02%             |

Table 2: Moody’s speculative grade cumulative default probabilities. From Exhibit 30, Keenan et al (2000).

<table>
<thead>
<tr>
<th>Year</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.35%</td>
</tr>
<tr>
<td>2</td>
<td>6.76%</td>
</tr>
<tr>
<td>3</td>
<td>9.98%</td>
</tr>
<tr>
<td>4</td>
<td>12.89%</td>
</tr>
<tr>
<td>5</td>
<td>15.57%</td>
</tr>
<tr>
<td>6</td>
<td>17.91%</td>
</tr>
</tbody>
</table>
Table 3: One year default statistics. Speculative grade.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>CreditMetrics Discrete</th>
<th>CreditMetrics Diffusion</th>
<th>CreditMetrics Copula</th>
<th>Stoch. Int. Slow</th>
<th>Stoch. Int. Fast</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low correlation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>3.37</td>
<td>3.36</td>
<td>3.51</td>
<td>3.20</td>
<td>3.20</td>
</tr>
<tr>
<td>St. Dev.</td>
<td>3.15</td>
<td>3.27</td>
<td>3.40</td>
<td>3.03</td>
<td>3.05</td>
</tr>
<tr>
<td>Median</td>
<td>3.00</td>
<td>2.98</td>
<td>3.01</td>
<td>2.98</td>
<td>2.98</td>
</tr>
<tr>
<td>5th percentile</td>
<td>10.00</td>
<td>9.99</td>
<td>10.01</td>
<td>9.99</td>
<td>9.99</td>
</tr>
<tr>
<td><strong>High correlation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>3.62</td>
<td>3.24</td>
<td>3.72</td>
<td>3.69</td>
<td>3.56</td>
</tr>
<tr>
<td>St. Dev.</td>
<td>7.08</td>
<td>6.32</td>
<td>7.52</td>
<td>6.84</td>
<td>6.73</td>
</tr>
<tr>
<td>Median</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>5th percentile</td>
<td>19.00</td>
<td>15.99</td>
<td>19.01</td>
<td>19.00</td>
<td>19.00</td>
</tr>
<tr>
<td>1st percentile</td>
<td>37.00</td>
<td>32.99</td>
<td>34.01</td>
<td>30.00</td>
<td>35.00</td>
</tr>
</tbody>
</table>

Table 4: Six year cumulative default statistics. Speculative grade.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>CreditMetrics Discrete</th>
<th>CreditMetrics Diffusion</th>
<th>CreditMetrics Copula</th>
<th>Stoch. Int. Slow</th>
<th>Stoch. Int. Fast</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low correlation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>17.72</td>
<td>16.93</td>
<td>18.04</td>
<td>17.34</td>
<td>18.10</td>
</tr>
<tr>
<td>St. Dev.</td>
<td>6.40</td>
<td>8.68</td>
<td>9.66</td>
<td>16.15</td>
<td>9.73</td>
</tr>
<tr>
<td>Median</td>
<td>17.00</td>
<td>16.00</td>
<td>17.00</td>
<td>12.00</td>
<td>16.00</td>
</tr>
<tr>
<td>5th percentile</td>
<td>29.00</td>
<td>33.00</td>
<td>37.00</td>
<td>52.00</td>
<td>37.00</td>
</tr>
<tr>
<td>1st percentile</td>
<td>34.00</td>
<td>42.00</td>
<td>47.00</td>
<td>73.00</td>
<td>49.00</td>
</tr>
<tr>
<td><strong>High correlation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>18.41</td>
<td>17.28</td>
<td>18.61</td>
<td>19.81</td>
<td>20.41</td>
</tr>
<tr>
<td>St. Dev.</td>
<td>13.49</td>
<td>17.41</td>
<td>19.27</td>
<td>24.37</td>
<td>19.36</td>
</tr>
<tr>
<td>Median</td>
<td>15.00</td>
<td>12.00</td>
<td>12.00</td>
<td>9.00</td>
<td>13.00</td>
</tr>
<tr>
<td>5th percentile</td>
<td>45.00</td>
<td>54.00</td>
<td>63.00</td>
<td>82.00</td>
<td>62.00</td>
</tr>
<tr>
<td>1st percentile</td>
<td>59.00</td>
<td>73.00</td>
<td>78.00</td>
<td>98.00</td>
<td>86.00</td>
</tr>
</tbody>
</table>
Table 5: Target expected losses for six year maturity. From Chart 3, Cifuentes et al (2000).

<table>
<thead>
<tr>
<th>Rating</th>
<th>Expected loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.002%</td>
</tr>
<tr>
<td>Aa1</td>
<td>0.023%</td>
</tr>
<tr>
<td>Aa2</td>
<td>0.048%</td>
</tr>
<tr>
<td>Aa3</td>
<td>0.101%</td>
</tr>
<tr>
<td>A1</td>
<td>0.181%</td>
</tr>
<tr>
<td>A2</td>
<td>0.320%</td>
</tr>
<tr>
<td>A3</td>
<td>0.500%</td>
</tr>
<tr>
<td>Baa1</td>
<td>0.753%</td>
</tr>
<tr>
<td>Baa2</td>
<td>1.083%</td>
</tr>
<tr>
<td>Baa3</td>
<td>2.035%</td>
</tr>
</tbody>
</table>

Table 6: Prices (in $M) for first and second loss protection. Expected loss, rating, and required capital ($M) for senior notes. Speculative grade collateral.

<table>
<thead>
<tr>
<th>Low correlation</th>
<th>First loss</th>
<th>Second loss</th>
<th>Exp. loss</th>
<th>Rating</th>
<th>Capital (Aa3)</th>
<th>Capital (A3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CM Discrete</td>
<td>7.227</td>
<td>1.350</td>
<td>0.000%</td>
<td>Aaa</td>
<td>17.3</td>
<td>13.8</td>
</tr>
<tr>
<td>CM Diffusion</td>
<td>6.676</td>
<td>1.533</td>
<td>0.017%</td>
<td>Aa1</td>
<td>21.6</td>
<td>15.9</td>
</tr>
<tr>
<td>Copula</td>
<td>6.788</td>
<td>1.936</td>
<td>0.022%</td>
<td>Aa1</td>
<td>24.5</td>
<td>18.0</td>
</tr>
<tr>
<td>Stoch. int. – slow</td>
<td>5.533</td>
<td>2.501</td>
<td>0.466%</td>
<td>A3</td>
<td>39.8</td>
<td>29.4</td>
</tr>
<tr>
<td>Stoch. int. – fast</td>
<td>6.763</td>
<td>1.911</td>
<td>0.038%</td>
<td>Aa2</td>
<td>25.7</td>
<td>18.3</td>
</tr>
<tr>
<td>High correlation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CM Discrete</td>
<td>6.117</td>
<td>2.698</td>
<td>0.159%</td>
<td>A1</td>
<td>32.3</td>
<td>23.6</td>
</tr>
<tr>
<td>CM Diffusion</td>
<td>5.144</td>
<td>2.832</td>
<td>0.514%</td>
<td>Baa1</td>
<td>41.1</td>
<td>30.2</td>
</tr>
<tr>
<td>Copula</td>
<td>5.210</td>
<td>3.200</td>
<td>0.821%</td>
<td>Baa2</td>
<td>43.7</td>
<td>34.4</td>
</tr>
<tr>
<td>Stoch. int. – slow</td>
<td>4.856</td>
<td>3.307</td>
<td>1.903%</td>
<td>Baa3</td>
<td>54.5</td>
<td>46.1</td>
</tr>
<tr>
<td>Stoch. int. – fast</td>
<td>5.685</td>
<td>3.500</td>
<td>0.918%</td>
<td>Baa2</td>
<td>45.9</td>
<td>35.2</td>
</tr>
</tbody>
</table>
Figure 1: Marginal and cumulative year two default volatility.

Investment grade, low correlation

Investment grade, high correlation

Speculative grade, low correlation

Speculative grade, high correlation
Figure 2: Year two conditional default probability given default of a second asset.

Investment grade, low correlation

Investment grade, high correlation

Speculative grade, low correlation

Speculative grade, high correlation
Figure 3: Normalized cumulative default rate volatilities. Speculative grade.

**Low correlation**

**High correlation**
Figure 4: Distribution of cumulative six year defaults. Speculative grade, low correlation.
Figure 5: Distribution of cumulative six year defaults, extreme cases. Speculative grade, low correlation.
Figure 6: Distribution of cumulative six year defaults. Speculative grade, high correlation.
Figure 7: Distribution of cumulative six year defaults, extreme cases. Speculative grade, high correlation.